

Relativistic Cosmological Perturbation Theory and the Evolution of Small-Scale Inhomogeneities

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(Dated: July 1, 2012)

It is shown that a first-order relativistic perturbation theory for the open, flat or closed Friedmann-Lemaître-Robertson-Walker universe admits one, and only one, gauge-invariant variable which describes the perturbation to the energy density and which becomes equal to the usual Newtonian energy density in the non-relativistic limit. The same holds true for the perturbation to the particle number density. Using these two new variables, a manifestly covariant and gauge-invariant cosmological perturbation theory, adapted to non-barotropic equations of state for the pressure, has been developed. The new perturbation theory is free from metric gradients.

Perturbations in the total energy density are gravitationally coupled to perturbations in the particle number density, irrespective of the nature of the particles. Small-scale perturbations in the radiation-dominated era oscillate with an increasing amplitude.

After decoupling of matter and radiation density perturbations evolve adiabatically, i.e., they exchange heat with their environment. This heat loss of a perturbation may enhance the growth rate of its mass sufficiently to explain stellar formation in the early universe, a phenomenon not understood, as yet, without the additional assumption of the existence of Cold Dark Matter. This theoretical observation is the main result of this article.

PACS numbers: 04.25.Nx, 04.20.Cv, 97.20.Wt, 98.80.Bp, 98.80.Jk

arXiv:1106.0627v3 [gr-qc] 2 Jul 2012

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I. INTRODUCTION

Since measurements of the fundamental parameters of our universe are very precise today, cosmology is nowadays a mature branch of astrophysics. Despite advances in observational as well as theoretical cosmology, there is, as yet, no consensus about which cosmological perturbation theory describes unequivocally the evolution of density perturbations.

A new approach to cosmological density perturbations is presented and evolution equations are given which are adapted to the situation that the equations of state are more realistic. The issue of metric gradients [1] does not exist in the new perturbation theory.

The calculations of Lifshitz [2] and Lifshitz and Khalatnikov [3] (see also Landau and Lifshitz [4], Chapter 14) are redone, and new results of the literature, which were not known at the time when Lifshitz and Khalatnikov developed their theory, are used. Combining these results with new insights described in this article, a perturbation theory for the open flat or closed Friedmann-Lemaître-Robertson-Walker (FLRW) universe is developed with the properties that both the evolution equations and their solutions are invariant under general infinitesimal coordinate transformations $x^\mu \rightarrow x^\mu - \xi^\mu(x^\nu)$. The Lifshitz-Khalatnikov cosmological perturbation theory has thus been rewritten into a *manifestly covariant and gauge-invariant* perturbation theory, see equations (43). The upshot of the new approach is a possible explanation of primeval stars, the so-called (hypothetical) population III stars, independent of the existence of Cold Dark Matter (CDM).

A. Former Insights

Bardeen was the first to point out the importance of gauge-invariant variables in the construction of a perturbation theory. In his seminal article [5] Bardeen demonstrated that the use of gauge-invariant variables ensures that a perturbation theory is free of spurious solutions. The article of Bardeen has inspired the pioneering works of Ellis *et al.* [6–8] and Mukhanov *et al.* [9, 10]. These researchers proposed alternative perturbation theories.

Although Lifshitz and Khalatnikov were aware of the fact that the system of perturbation equations can be divided into three independent systems of equations, namely for tensor perturbations (gravitational waves), vector perturbations (vortices), and scalar perturbations, they did not explicitly use the decompositions (17) and (20) found by York [11], Stewart [12] and Stewart and Walker [13]. These decompositions make the computations much more tractable.

An important result [11–13] is that the perturbed metric tensor for scalar perturbations can be written in terms of two potentials (19). This facilitates the derivation of the Newtonian results (66) and (67) in the non-relativistic limit.

B. New Insights

In addition to the results found in the literature, some new insights, described in this article, are used.

Firstly, barotropic equations of state for the pressure $p = p(\varepsilon)$, where ε is the energy density, are commonly used in cosmology [14]. In particular, the linear barotropic equation of state $p = w\varepsilon$, with $w = \frac{1}{3}$ in the radiation-dominated era and $w = 0$ in the era after decoupling of matter and radiation, describes very well the global evolution of an *unperturbed* FLRW universe. As will become clear in Section VI, $p = w\varepsilon$ gives an incomplete description of the evolution of density perturbations in the *perturbed* universe. Therefore, realistic equations of state are needed. From thermodynamics it is known that both the energy density ε and the pressure p depend on the independent quantities n and T , i.e.,

$$\varepsilon = \varepsilon(n, T), \quad p = p(n, T), \quad (1)$$

where n is the particle number density and T the temperature. Since T can, in principle, be eliminated from these equations of state, a computationally more convenient equation of state for the pressure will be used, namely

$$p = p(n, \varepsilon). \quad (2)$$

This, non-barotropic, equation of state has been included in the new perturbation theory (43). Therefore, this theory will be referred to as the *generalised Lifshitz-Khalatnikov* theory.

Secondly, for scalar perturbations the three perturbed momentum constraint equations (10b) can be rewritten as one evolution equation (28b) for the local perturbation $R_{(1)}$, (15), to the global spatial curvature $R_{(0)}$, (8). Since, in first-order, only the *irrotational* part of the spatial part of the fluid four-velocity is coupled to scalar perturbations, the three momentum conservation laws (10e) can be recast in one evolution equation (28d) for the divergence $\vartheta_{(1)}$,

(16), of the spatial part of the fluid four-velocity. Using the variables $R_{(1)}$ and $\vartheta_{(1)}$, the evolution equations for scalar perturbations can be written as an initial value problem (28) with four *ordinary* differential equations and one algebraic equation, to be obeyed by the initial values. The only gradient term in this system occurs in equation (28d). This is a distinct advantage over former perturbation theories [1, 5–10] which contain metric gradients.

Thirdly, since the sets of equations (7) and (28) are written with respect to the *same* system of reference, one can draw the conclusion that only three independent scalars, namely the energy density ε , the particle number density n and the expansion scalar θ , play a role in a perturbation theory. This reduces the number of possible gauge-invariant variables considerably, and, thereupon, leads to the insight that there exist two, and only two, unique gauge-invariant quantities $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$, defined by (39a).

Finally, in order to show that $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ are indeed the perturbations to the energy density and particle number density, respectively, one has to investigate the basic equations (28) for scalar perturbations together with the definitions (39a) in the so-called *non-relativistic limit*. In this limit, which will be defined in exact terms in Section III H, the set (28) combined with the definitions (39a) reduce to the Newtonian Theory of Gravity, i.e., the Newtonian results (66) and (67) show up. Therefore, $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ are the real, physical, energy density perturbation and the particle number density perturbation, respectively. Consequently, the generalised Lifshitz-Khalatnikov theory (43), which is derived from (28) using (39a), is a manifestly covariant and gauge-invariant perturbation theory and describes unequivocally the evolution of density perturbations in FLRW universes.

II. RESULTS

Firstly, it follows from the generalised Lifshitz-Khalatnikov theory that perturbations in the total energy density are *gravitationally* coupled to perturbations in the particle number density (43b), irrespective of the nature of the particles (i.e., ordinary matter or CDM) and independent of the scale of the perturbations.

Secondly, in the radiation-dominated era of a flat FLRW universe small-scale perturbations oscillate with an *increasing* amplitude according to (76), whereas the standard evolution equation (104) predicts oscillatory perturbations with a *constant* amplitude. This difference is due to the fact that in the new perturbation theory the divergence $\vartheta_{(1)}$, (16), of the spatial part of the fluid four-velocity has been taken into account, whereas $\vartheta_{(1)} = 0$ in the standard theory, compare the Newtonian equation (104) with the exact relativistic equations (107). In the evolution of large-scale perturbations $\vartheta_{(1)}$ plays only a minor role, so that the results of the standard theory are recovered (75). Heat exchange does not play a role in the radiation-dominated era.

Finally, in the era after decoupling of matter and radiation, heat loss, Section III G, of a perturbation, in addition to gravity, has a more or less favourable effect on its growth rate, depending on the scale. For large-scale perturbations heat loss is unimportant and gravity is the main cause of contraction. Therefore, the generalised Lifshitz-Khalatnikov theory corroborates the results for large-scale perturbations (88) of former perturbation theories which did not take heat loss into account. Small-scale perturbations benefit more from heat loss during their evolution than large-scale perturbations do. It is shown in Section V C that for perturbations with scales around 6.2 pc gravity and heat loss combine optimally, Figure 1, resulting in fast growing density perturbations. This may shed new light on the evolution of small-scale inhomogeneities in the universe.

III. GENERALISED LIFSHITZ-KHALATNIKOV PERTURBATION THEORY

In this section the generalised Lifshitz-Khalatnikov theory of cosmological density perturbations is derived for the open, flat or closed FLRW universe. To that end, a suitable system of reference must be chosen.

Due to the general covariance of the Einstein equations and conservation laws, Einstein's gravitational theory is invariant under a general coordinate transformation $x^\mu \rightarrow \hat{x}^\mu(x^\nu)$, implying that preferred coordinate systems do not exist (Weinberg [15], Appendix B). In particular, the linearised Einstein equations and conservation laws are invariant under a general linear coordinate transformation

$$x^0 \rightarrow x^0 - \psi(t, \mathbf{x}), \quad x^i \rightarrow x^i - \xi^i(t, \mathbf{x}), \quad (3)$$

where $\psi(t, \mathbf{x})$ and $\xi^i(t, \mathbf{x})$ are four arbitrary, first-order (infinitesimal) functions of the time ($x^0 = ct$) and space [$\mathbf{x} = (x^1, x^2, x^3)$] coordinates. Since preferred systems of reference do not exist and since the final result (43) is manifestly covariant and gauge-invariant, one may use any suitable and convenient coordinate system to perform the calculations. In order to put an accurate interpretation on the new gauge-invariant quantities (39) one needs the non-relativistic limit. In the Newtonian Theory of Gravity all possible systems of reference are *synchronous*. Therefore, synchronous coordinates [2–4], i.e., $g_{0i} = 0$, will be used in the background as well as in the perturbed universe. For convenience, synchronous coordinates are chosen such that coordinate time is equal to proper time, i.e., $g_{00} = 1$.

A. Basic Equations in Synchronous Coordinates

In synchronous coordinates the metric of FLRW universes has the form

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{ij} = -a^2(t)\tilde{g}_{ij}(\mathbf{x}), \quad (4)$$

where $a(t)$ is the scale factor of the universe, and \tilde{g}_{ij} is the metric of the three-dimensional maximally symmetric subspaces of constant time. The functions ψ and ξ^i of the general infinitesimal transformation (3) reduce to

$$\psi = \psi(\mathbf{x}), \quad \xi^i = \tilde{g}^{ik}\partial_k\psi(\mathbf{x}) \int^{ct} \frac{d\tau}{a^2(\tau)} + \chi^i(\mathbf{x}), \quad (5)$$

if only transformations between synchronous coordinates are allowed. In (5), $\psi(\mathbf{x})$ and $\chi^i(\mathbf{x})$ are four arbitrary, infinitesimal functions of the spatial coordinates.

1. Background Equations

The complete set of zeroth-order Einstein equations and conservation laws for an open, flat or closed FLRW universe filled with a perfect fluid with energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}, \quad p = p(n, \varepsilon), \quad (6)$$

is, in synchronous coordinates, given by

$$3H^2 = \frac{1}{2}R_{(0)} + \kappa\varepsilon_{(0)} + \Lambda, \quad \kappa = 8\pi G_N/c^4, \quad (7a)$$

$$\dot{R}_{(0)} = -2HR_{(0)}, \quad (7b)$$

$$\dot{\varepsilon}_{(0)} = -3H\varepsilon_{(0)}(1 + w), \quad w \equiv p_{(0)}/\varepsilon_{(0)}, \quad (7c)$$

$$\vartheta_{(0)} = 0, \quad (7d)$$

$$\dot{n}_{(0)} = -3Hn_{(0)}. \quad (7e)$$

The G_{0i} constraint equations and the G_{ij} , $i \neq j$, dynamical equations are identically satisfied. The G_{ii} dynamical equations are equivalent to the time-derivative of the G_{00} constraint equation (7a). Therefore, the G_{ij} dynamical equations need not be taken into account. In equations (7) Λ is the cosmological constant, G_N the gravitational constant of the Newtonian Theory of Gravity, c the speed of light and w a shorthand notation for the quotient $p_{(0)}/\varepsilon_{(0)}$. An over-dot denotes differentiation with respect to ct and the sub-index (0) refers to the background, i.e., unperturbed, quantities. Furthermore, $H \equiv \dot{a}/a$ is the Hubble function which is equal to $H = \frac{1}{3}\theta_{(0)}$, where $\theta_{(0)}$ is the background value of the expansion scalar $\theta \equiv u^\mu{}_{;\mu}$ with $u^\mu \equiv c^{-1}U^\mu$ the four-velocity, normalised to unity ($u^\mu u_\mu = 1$). A semicolon denotes covariant differentiation with respect to the background metric $g_{(0)\mu\nu}$. The *spatial* part of the background Ricci curvature tensor $R^i_{(0)j}$ and its trace $R_{(0)}$ are given by

$$R^i_{(0)j} = -\frac{2K}{a^2}\delta^i_j, \quad R_{(0)} = -\frac{6K}{a^2}, \quad K = -1, 0, +1, \quad (8)$$

where $R_{(0)}$ is the global spatial curvature. The quantity $\vartheta_{(0)}$ is the three-divergence of the spatial part of the four-velocity $u^\mu_{(0)}$. For an isotropically expanding universe the four-velocity is $u^\mu_{(0)} = \delta^\mu_0$, so that $\vartheta_{(0)} = 0$.

From the system (7) one may infer that the evolution of an unperturbed FLRW universe is determined by exactly three independent scalars, namely

$$\varepsilon = T^{\mu\nu}u_\mu u_\nu, \quad n = N^\mu u_\mu, \quad \theta = u^\mu{}_{;\mu}, \quad (9)$$

where $N^\mu \equiv nu^\mu$ is the cosmological particle current four-vector, which satisfies the particle number conservation law $N^\mu{}_{;\mu} = 0$, (7e), see Weinberg [15], Appendix B. As will become clear in Section III D, the quantities (9) and their first-order counterparts play a key role in the evolution of cosmological density perturbations.

2. Perturbation Equations

The complete set of first-order Einstein equations and conservation laws for the open, flat or closed FLRW universe is, in synchronous coordinates, given by

$$H\dot{h}^k_k + \frac{1}{2}R_{(1)} = -\kappa\varepsilon_{(1)}, \quad (10a)$$

$$\dot{h}^k_{k|i} - \dot{h}^k_{i|k} = 2\kappa(\varepsilon_{(0)} + p_{(0)})u_{(1)i}, \quad (10b)$$

$$\ddot{h}^i_j + 3H\dot{h}^i_j + \delta^i_j H\dot{h}^k_k + 2R^i_{(1)j} = -\kappa\delta^i_j(\varepsilon_{(1)} - p_{(1)}), \quad (10c)$$

$$\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + (\varepsilon_{(0)} + p_{(0)})\theta_{(1)} = 0, \quad (10d)$$

$$\frac{1}{c} \frac{d}{dt} \left[(\varepsilon_{(0)} + p_{(0)})u^i_{(1)} \right] - g^{ik}_{(0)}p_{(1)|k} + 5H(\varepsilon_{(0)} + p_{(0)})u^i_{(1)} = 0, \quad (10e)$$

$$\dot{n}_{(1)} + 3Hn_{(1)} + n_{(0)}\theta_{(1)} = 0, \quad (10f)$$

where $h_{\mu\nu} \equiv -g_{(1)\mu\nu}$ with $h_{00} = 0$, $h_{0i} = 0$ is the perturbed metric, $h^i_j = g^{ik}_{(0)}h_{kj}$, and $g^{ij}_{(0)}$ is the unperturbed background metric (4) for an open, flat or closed FLRW universe. Quantities with a sub-index (1) are the first-order counterparts of the background quantities with a sub-index (0). A vertical bar denotes covariant differentiation with respect to $g_{(0)ij}$.

The first-order perturbation to the pressure is given by the perturbed equation of state

$$p_{(1)} = p_n n_{(1)} + p_\varepsilon \varepsilon_{(1)}, \quad p_n \equiv \left(\frac{\partial p}{\partial n} \right)_\varepsilon, \quad p_\varepsilon \equiv \left(\frac{\partial p}{\partial \varepsilon} \right)_n, \quad (11)$$

where $p_n(n, \varepsilon)$ and $p_\varepsilon(n, \varepsilon)$ are the partial derivatives of the equation of state $p(n, \varepsilon)$.

The first-order perturbation to the spatial part of the Ricci tensor (8) reads

$$R^i_{(1)j} \equiv (g^{ik}R_{kj})_{(1)} = g^{ik}_{(0)}R_{(1)kj} + \frac{1}{3}R_{(0)}h^i_j. \quad (12)$$

Using Lifshitz' formula [see Weinberg [16], equation (10.9.1)]

$$\Gamma^k_{(1)ij} = -\frac{1}{2}g^{kl}_{(0)}(h_{li|j} + h_{lj|i} - h_{ij|l}), \quad (13)$$

and the contracted Palatini identities [see Weinberg [16], equation (10.9.2)]

$$R_{(1)ij} = \Gamma^k_{(1)ij|k} - \Gamma^k_{(1)ik|j}, \quad (14)$$

one finds, using $g^{ij}_{(0)}h^k_{i|j|k} = g^{ij}_{(0)}h^k_{i|k|j}$, for the trace of (12)

$$R_{(1)} = g^{ij}_{(0)}(h^k_{k|i|j} - h^k_{i|k|j}) + \frac{1}{3}R_{(0)}h^k_k. \quad (15)$$

Expression (15) is the local perturbation to the global spatial curvature $R_{(0)}$ due to a local density perturbation.

Finally, $\theta_{(1)}$ is the first-order perturbation to the expansion scalar $\theta \equiv u^\mu_{;\mu}$. Using that $u^\mu_{(0)} = \delta^\mu_0$, one gets

$$\theta_{(1)} = \vartheta_{(1)} - \frac{1}{2}\dot{h}^k_k, \quad \vartheta_{(1)} \equiv u^k_{(1)|k}, \quad (16)$$

where $\vartheta_{(1)}$ is the divergence of the spatial part of the perturbed four-velocity $u^\mu_{(1)}$. The quantities (15) and (16) play an important role in the derivation of the manifestly covariant and gauge-invariant perturbation theory (43).

B. Decomposition of the Metric and the Spatial Part of the Fluid Four-Velocity

Since the work of Lifshitz and Khalatnikov, it is known that for FLRW universes the set (10) can be broken down into three independent systems of equations, namely a system for gravitational waves (tensor perturbations), for vortices (vector perturbations), and for scalar perturbations. In addition they showed that only scalar perturbations are coupled to density perturbations. Lifshitz and Khalatnikov used spherical harmonics to classify the three types of perturbations. A disadvantage of their approach is that using spherical harmonics is computationally demanding. A new insight, gained in 1974 by York [11], and in 1990 elucidated by Stewart [12], makes the computations much

more tractable. York and Stewart showed that the symmetric perturbation tensor h_{ij} can uniquely be decomposed into three parts, i.e.,

$$h^i_j = h^i_{||j} + h^i_{\perp j} + h^i_{*j}, \quad (17)$$

where the scalar, vector and tensor perturbations are denoted by $||$, \perp and $*$, respectively. The constituents have the properties

$$h^k_{\perp k} = 0, \quad h^k_{*k} = 0, \quad h^k_{*i|k} = 0. \quad (18)$$

Moreover, York and Stewart demonstrated that the components $h^i_{||j}$ can be written in terms of two independent potentials $\phi(t, \mathbf{x})$ and $\zeta(t, \mathbf{x})$, namely

$$h^i_{||j} = \frac{2}{c^2}(\phi \delta^i_j + \zeta^{|i}_{|j}). \quad (19)$$

Finally, Stewart also proved that the spatial part of the perturbed four-velocity $\mathbf{u}_{(1)}$ can uniquely be decomposed into two parts

$$\mathbf{u}_{(1)} = \mathbf{u}_{(1)||} + \mathbf{u}_{(1)\perp}, \quad (20)$$

where the constituents have the properties

$$\tilde{\nabla} \cdot \mathbf{u}_{(1)} = \tilde{\nabla} \cdot \mathbf{u}_{(1)||}, \quad \tilde{\nabla} \times \mathbf{u}_{(1)} = \tilde{\nabla} \times \mathbf{u}_{(1)\perp}, \quad (21)$$

with $\tilde{\nabla}^i \equiv \tilde{g}^{ij} \partial_j$, the generalised vector differential operator.

The three different kinds of perturbations will now be considered according to the decompositions (17) and (20), and their properties (18) and (21).

For tensor perturbations the properties (18) imply that $R_{(1)*} = 0$, as follows from (15). Using this and (18), equations (10a)–(10c) imply that tensor perturbations are not coupled to $\varepsilon_{(1)}$, $p_{(1)}$ and $\mathbf{u}_{(1)}$.

The perturbed Ricci tensor, $R_{(1)ij}$, being a symmetric tensor, should obey the decomposition (17) with the properties (18), namely $R^k_{(1)\perp k} = 0$. This implies that, see (15), h^i_{\perp} must obey $h^{kl}_{\perp|k|l} = 0$, in addition to $h^k_{\perp k} = 0$. From (10a) and the trace of (10c) it follows that vector perturbations are not coupled to $\varepsilon_{(1)}$ and $p_{(1)}$. Raising the index i of equation (10b) with $g^{ij}_{(0)}$, one finds that this equation reads for vector perturbations

$$\dot{h}^{kj}_{\perp|k} + 2H h^{kj}_{\perp|k} = 2\kappa(\varepsilon_{(0)} + p_{(0)})u^j_{(1)}, \quad (22)$$

where it is used that $\dot{g}^{ij}_{(0)} = -2H g^{ij}_{(0)}$. Taking the covariant derivative of (22) with respect to the index j one finds with the additional property $h^{kl}_{\perp|k|l} = 0$ that equation (22) reduces to $u^j_{(1)|j} = 0$, implying with (21) that the constituent $\mathbf{u}_{(1)\perp}$ is coupled to vector perturbations.

Since both $h^k_{||k} \neq 0$ and $R_{(1)||} \neq 0$, scalar perturbations are coupled to $\varepsilon_{(1)}$ and $p_{(1)}$. It will now be demonstrated that $\mathbf{u}_{(1)||}$ is coupled to scalar perturbations, by showing that equation (10b) requires that the rotation of $\mathbf{u}_{(1)}$ vanishes, if the metric is of the form (19). Differentiating (10b) covariantly with respect to the index j and substituting (19) yields

$$2\dot{\phi}_{|i|j} + \dot{\zeta}^{|k}_{|k|j} - \dot{\zeta}^{|k}_{|i|k|j} = \kappa c^2(\varepsilon_{(0)} + p_{(0)})u_{(1)i|j}. \quad (23)$$

Interchanging i and j and subtracting the result from (23) one gets

$$\dot{\zeta}^{|k}_{|i|k|j} - \dot{\zeta}^{|k}_{|j|k|i} = -\kappa c^2(\varepsilon_{(0)} + p_{(0)})(u_{(1)i|j} - u_{(1)j|i}). \quad (24)$$

By rearranging the covariant derivatives, (24) can be cast in the form

$$\begin{aligned} &(\dot{\zeta}^{|k}_{|i|k|j} - \dot{\zeta}^{|k}_{|j|k|i}) - (\dot{\zeta}^{|k}_{|j|k|i} - \dot{\zeta}^{|k}_{|i|k|j}) \\ &+ (\dot{\zeta}^{|k}_{|i|j} - \dot{\zeta}^{|k}_{|j|i})_{|k} = -\kappa c^2(\varepsilon_{(0)} + p_{(0)})(u_{(1)i|j} - u_{(1)j|i}). \end{aligned} \quad (25)$$

Using the expressions for the commutator of second order covariant derivatives (Weinberg [16], Chapter 6, Section 5)

$$A^i_{j|p|q} - A^i_{j|q|p} = A^i_k R^k_{(0)jppq} - A^k_j R^i_{(0)kpq}, \quad B^i_{|p|q} - B^i_{|q|p} = B^k R^i_{(0)kpq}, \quad (26)$$

and substituting the background Riemann tensor for three-spaces of constant time

$$R^i_{(0)jkl} = K(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}), \quad K = -1, 0, +1, \quad (27)$$

one finds that the left-hand side of equation (25) vanishes identically, implying that the rotation of $\mathbf{u}_{(1)}$ is zero. Therefore, only $\mathbf{u}_{(1)||}$ is coupled to scalar perturbations.

C. First-order Equations for Scalar Perturbations

Since scalar perturbations, i.e., perturbations in $\varepsilon_{(1)}$ and $p_{(1)}$, are only coupled to $h_{||j}^i$ and $u_{(1)||}^i$, one may replace in (10)–(16) $h_{||j}^i$ by $h_{||j}^i$ and $u_{(1)}^i$ by $u_{(1)||}^i$, to obtain perturbation equations which exclusively describe the evolution of scalar perturbations. From now on, only scalar perturbations are considered, and the subscript $||$ will be omitted. Using the decompositions (17) and (20) and the properties (18) and (21), one can rewrite the evolution equations for scalar perturbations in the form

$$2H(\theta_{(1)} - \vartheta_{(1)}) - \frac{1}{2}R_{(1)} = \kappa\varepsilon_{(1)}, \quad (28a)$$

$$\dot{R}_{(1)} + 2HR_{(1)} - 2\kappa\varepsilon_{(0)}(1+w)\vartheta_{(1)} + \frac{2}{3}R_{(0)}(\theta_{(1)} - \vartheta_{(1)}) = 0, \quad (28b)$$

$$\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)}(1+w)\theta_{(1)} = 0, \quad (28c)$$

$$\dot{\vartheta}_{(1)} + H(2 - 3\beta^2)\vartheta_{(1)} + \frac{1}{\varepsilon_{(0)}(1+w)} \frac{\tilde{\nabla}^2 p_{(1)}}{a^2} = 0, \quad \beta^2 \equiv \frac{\dot{p}_{(0)}}{\dot{\varepsilon}_{(0)}}, \quad (28d)$$

$$\dot{n}_{(1)} + 3Hn_{(1)} + n_{(0)}\theta_{(1)} = 0. \quad (28e)$$

The set (28), which is the perturbed counterpart of the set (7), consists of one algebraic equation (28a) and four ordinary differential equations (28b)–(28e) for the five unknown quantities $\varepsilon_{(1)}$, $n_{(1)}$, $\vartheta_{(1)}$, $R_{(1)}$ and $\theta_{(1)}$. The quantity $\beta(t)$ is defined by $\beta^2 \equiv \dot{p}_{(0)}/\dot{\varepsilon}_{(0)}$. Using that $\dot{p}_{(0)} = p_n \dot{n}_{(0)} + p_\varepsilon \dot{\varepsilon}_{(0)}$ and the conservation laws (7c) and (7e) one gets

$$\beta^2 = p_\varepsilon + \frac{n_{(0)}p_n}{\varepsilon_{(0)}(1+w)}. \quad (29)$$

Finally, the symbol $\tilde{\nabla}^2$ denotes the generalised Laplace operator with respect to the three-space metric \tilde{g}_{ij} , defined by $\tilde{\nabla}^2 f \equiv \tilde{g}^{ij} f_{|i|j}$.

The derivation of the basic equations (28) for scalar perturbations will now be given. Eliminating $\dot{h}_{||k}^k$ from (10a) with the help of (16) yields the algebraic equation (28a).

Multiplying both sides of equation (10b) by $g_{(0)}^{ij}$ and taking the covariant derivative with respect to the index j , one finds

$$g_{(0)}^{ij}(\dot{h}_{||k|j}^k - \dot{h}_{||i|k|j}^k) = 2\kappa(\varepsilon_{(0)} + p_{(0)})\vartheta_{(1)}, \quad (30)$$

where also (16) has been used. The left-hand side of (30) will turn up as a part of the time-derivative of the curvature $R_{(1)}$. In fact, differentiating (15) with respect to time and recalling that the background connection coefficients $\Gamma_{(0)ij}^k$ are for FLRW metrics (4) independent of time, one gets, using also $\dot{g}_{(0)}^{ij} = -2Hg_{(0)}^{ij}$ and (7b),

$$\dot{R}_{(1)} = -2HR_{(1)} + g_{(0)}^{ij}(\dot{h}_{||k|j}^k - \dot{h}_{||i|k|j}^k) + \frac{1}{3}R_{(0)}\dot{h}_{||k}^k. \quad (31)$$

Combining (30) and (31) and using (16) to eliminate $\dot{h}_{||k}^k$ yields (28b). Thus, the $G_{(1)i}^0$ momentum constraint equations (10b) have been recast into one equation (28b) for the local spatial curvature due to a density perturbation.

For $i \neq j$ equations (10c) need not be considered, since they are not coupled to scalar perturbations. Taking the trace of (10c) and eliminating the quantity $\dot{h}_{||k}^k$ with the help of (16), one arrives at

$$\dot{\theta}_{(1)} - \dot{\vartheta}_{(1)} + 6H(\theta_{(1)} - \vartheta_{(1)}) - R_{(1)} = \frac{3}{2}\kappa(\varepsilon_{(1)} - p_{(1)}). \quad (32)$$

Using (28a) to eliminate the second term of (32) yields for the trace of (10c)

$$\dot{\theta}_{(1)} - \dot{\vartheta}_{(1)} + \frac{1}{2}R_{(1)} = -\frac{3}{2}\kappa(\varepsilon_{(1)} + p_{(1)}). \quad (33)$$

This equation is equal to the time-derivative of the constraint equation (28a), which reads

$$2\dot{H}(\theta_{(1)} - \vartheta_{(1)}) + 2H(\dot{\theta}_{(1)} - \dot{\vartheta}_{(1)}) - \frac{1}{2}\dot{R}_{(1)} = \kappa\dot{\varepsilon}_{(1)}. \quad (34)$$

Eliminating the time-derivatives \dot{H} , $\dot{R}_{(1)}$ and $\dot{\varepsilon}_{(1)}$ with the help of (7a)–(7c), (28b) and (28c), respectively, yields the dynamical equation (33). Consequently, for scalar perturbations the dynamical equations (10c) need not be considered.

Finally, taking the covariant derivative of (10e) with respect to the metric $g_{(0)ij}$ and using (16), one gets

$$\frac{1}{c} \frac{d}{dt} \left[(\varepsilon_{(0)} + p_{(0)})\vartheta_{(1)} \right] - g_{(0)}^{ik} p_{(1)|k|i} + 5H(\varepsilon_{(0)} + p_{(0)})\vartheta_{(1)} = 0, \quad (35)$$

where it is used that the operations of taking the time-derivative and the covariant derivative commute, since the background connection coefficients $\Gamma_{(0)ij}^k$ are independent of time for FLRW metrics. With (4), $\tilde{\nabla}^2 f \equiv \tilde{g}^{ij} f_{|i|j}$ and (7c) one can rewrite (35) in the form

$$\dot{\vartheta}_{(1)} + H \left(2 - 3 \frac{\dot{p}_{(0)}}{\dot{\varepsilon}_{(0)}} \right) \vartheta_{(1)} + \frac{1}{\varepsilon_{(0)} + p_{(0)}} \frac{\tilde{\nabla}^2 p_{(1)}}{a^2} = 0, \quad (36)$$

Using the definitions $w \equiv p_{(0)}/\varepsilon_{(0)}$ and $\beta^2 \equiv \dot{p}_{(0)}/\dot{\varepsilon}_{(0)}$ one arrives at equation (28d).

This concludes the derivation of the system (28). As follows from its derivation, this system is, for scalar perturbations, equivalent to the full set of first-order Einstein equations and conservation laws (10).

D. Unique Gauge-invariant Cosmological Density Perturbations

The background equations (7) and the perturbation equations (28) are both written with respect to the *same* system of reference. Therefore, these two sets can be combined to describe the evolution of the five background quantities $\theta_{(0)} = 3H$, $R_{(0)}$, $\varepsilon_{(0)}$, $\vartheta_{(0)} = 0$, $n_{(0)}$, and their first-order counterparts $\theta_{(1)}$, $R_{(1)}$, $\varepsilon_{(1)}$, $\vartheta_{(1)}$, $n_{(1)}$. Just as in the background case, one again comes across the three independent scalars (9). Consequently, the evolution of cosmological density perturbations is described by the three *independent* scalars (9). A complicating factor is that the first-order quantities $\varepsilon_{(1)}$ and $n_{(1)}$, which are supposed to describe the energy density and the particle number density perturbations, have no physical significance, as will now be established.

A first-order perturbation to one of the scalars (9) transforms under a general (not necessarily between synchronous coordinates) infinitesimal coordinate transformation (3) as

$$S_{(1)}(t, \mathbf{x}) \rightarrow S_{(1)}(t, \mathbf{x}) + \psi(t, \mathbf{x}) \dot{S}_{(0)}(t), \quad (37)$$

where $S_{(0)}$ and $S_{(1)}$ are the background and first-order perturbation of one of the three scalars $S = \varepsilon, n, \theta$. In (37) the term $\dot{S} \equiv \psi \dot{S}_{(0)}$ is the so-called gauge mode. The complete set of gauge modes for the system of equations (28) is given by

$$\hat{\varepsilon}_{(1)} = \psi \dot{\varepsilon}_{(0)}, \quad \hat{n}_{(1)} = \psi \dot{n}_{(0)}, \quad \hat{\theta}_{(1)} = \psi \dot{\theta}_{(0)}, \quad (38a)$$

$$\hat{\vartheta}_{(1)} = -\frac{\tilde{\nabla}^2 \psi}{a^2}, \quad \hat{R}_{(1)} = 4H \left[\frac{\tilde{\nabla}^2 \psi}{a^2} - \frac{1}{2} R_{(0)} \psi \right], \quad (38b)$$

where $\psi = \psi(\mathbf{x})$ in synchronous coordinates, see (5). The quantities (38) are mere coordinate artifacts, which have no physical meaning, since the gauge function $\psi(\mathbf{x})$ is an arbitrary (infinitesimal) function. Equations (28) are invariant under coordinate transformations (3) with (5), i.e., the gauge modes (38) are solutions of the set (28). This property combined with the *linearity* of the perturbation equations, implies that a solution set $(\varepsilon_{(1)}, n_{(1)}, \theta_{(1)}, \vartheta_{(1)}, R_{(1)})$ can be augmented with the corresponding gauge modes (38) to obtain a new solution set. Therefore, the solution set $(\varepsilon_{(1)}, n_{(1)}, \theta_{(1)}, \vartheta_{(1)}, R_{(1)})$ has no physical significance, since the general solution of the set (28) can be modified by an infinitesimal coordinate transformation. This is the notorious gauge problem of cosmology.

In this article, the cosmological gauge problem has been resolved. To that end, the perturbation equations (10) have first been rewritten into the form (28) in order to isolate the scalar perturbations from the vortices and gravitational waves. This reduces the number of possible gauge-invariant variables substantially, since one needs to consider only the three independent scalars (9). All scalar perturbations transform under the general infinitesimal transformation (3) in the same way according to (37). This implies that one can combine two independent scalars to eliminate the gauge function $\psi(t, \mathbf{x})$. With the three independent scalars (9), one can make $\binom{3}{2} = 3$ different sets of three gauge-invariant variables. In each of these sets exactly one gauge-invariant quantity vanishes. As will be shown in Section III H, the only set for which the corresponding perturbation theory yields the Newtonian results (66) and (67) in the non-relativistic limit is given by

$$\varepsilon_{(1)}^{\text{gi}} = \varepsilon_{(1)} - \frac{\dot{\varepsilon}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)}, \quad n_{(1)}^{\text{gi}} = n_{(1)} - \frac{\dot{n}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)}, \quad (39a)$$

$$\theta_{(1)}^{\text{gi}} = \theta_{(1)} - \frac{\dot{\theta}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)} \equiv 0. \quad (39b)$$

It follows from the general transformation rule (37) that the quantities (39) are invariant under the general infinitesimal transformation (3), i.e., they are *gauge-invariant*, hence the superscript ‘gi’. The definitions (39) imply that the gauge-invariant counterpart $\theta_{(1)}^{\text{gi}}$ of the gauge dependent variable $\theta_{(1)} \neq 0$ vanishes automatically. The physical interpretation of (39b) is that, in first-order, the *global* expansion $\theta_{(0)} = 3H$ is not affected by a *local* density perturbation.

The quantities (39a) have two essential properties, which gauge-invariant quantities used in former perturbation theories [5–10] do not have. Firstly, due to the quotient $\dot{S}_{(0)}/\dot{\theta}_{(0)}$ of time derivatives, the definitions (39) are *independent* of the definition of time. As a consequence, the evolution of $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ will only be determined by their propagation equations. Secondly, the quantities (39a) do not contain *spatial* derivatives, so that unnecessary gradients [1] do not occur in the final equations (43).

The quantities (39a) are completely determined by the background equations (7) and their perturbed counterparts (28). In principle, these two sets can be used to study the evolution of density perturbations in FLRW universes. However, the set (28) is still too complicated, since it also admits the non-physical solutions (38). The aim will be a system of equations for $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ that do not have the gauge modes (38) as solution. In other words, a perturbation theory will be constructed for which not only the differential equations are invariant under general infinitesimal coordinate transformations (3), but also their solutions. Such a theory will be referred to as a *manifestly covariant and gauge-invariant* perturbation theory. The construction of this theory will be the subject of the next subsection.

E. Manifestly Covariant and Gauge-invariant Perturbation Theory

In this section the derivation of the new perturbation theory is given. Firstly, it is observed that the gauge dependent variable $\theta_{(1)}$ is not needed in the calculations, since its gauge-invariant counterpart $\theta_{(1)}^{\text{gi}}$, (39b), vanishes identically. Eliminating $\theta_{(1)}$ from the differential equations (28b)–(28e) with the help of the (algebraic) constraint equation (28a) yields the set of four first-order ordinary differential equations

$$\dot{R}_{(1)} + 2HR_{(1)} - 2\kappa\varepsilon_{(0)}(1+w)\vartheta_{(1)} + \frac{R_{(0)}}{3H}(\kappa\varepsilon_{(1)} + \frac{1}{2}R_{(1)}) = 0, \quad (40a)$$

$$\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)}(1+w)\left[\vartheta_{(1)} + \frac{1}{2H}(\kappa\varepsilon_{(1)} + \frac{1}{2}R_{(1)})\right] = 0, \quad (40b)$$

$$\dot{\vartheta}_{(1)} + H(2 - 3\beta^2)\vartheta_{(1)} + \frac{1}{\varepsilon_{(0)}(1+w)}\frac{\tilde{\nabla}^2 p_{(1)}}{a^2} = 0, \quad (40c)$$

$$\dot{n}_{(1)} + 3Hn_{(1)} + n_{(0)}\left[\vartheta_{(1)} + \frac{1}{2H}(\kappa\varepsilon_{(1)} + \frac{1}{2}R_{(1)})\right] = 0, \quad (40d)$$

for the four quantities $\varepsilon_{(1)}$, $n_{(1)}$, $\vartheta_{(1)}$ and $R_{(1)}$.

Using the background equations (7) to eliminate all time-derivatives and the first-order constraint equation (28a) to eliminate $\theta_{(1)}$, the gauge-invariant quantities (39a) become

$$\varepsilon_{(1)}^{\text{gi}} = \frac{\varepsilon_{(1)}R_{(0)} - 3\varepsilon_{(0)}(1+w)(2H\vartheta_{(1)} + \frac{1}{2}R_{(1)})}{R_{(0)} + 3\kappa\varepsilon_{(0)}(1+w)}, \quad (41a)$$

$$n_{(1)}^{\text{gi}} = n_{(1)} - \frac{3n_{(0)}(\kappa\varepsilon_{(1)} + 2H\vartheta_{(1)} + \frac{1}{2}R_{(1)})}{R_{(0)} + 3\kappa\varepsilon_{(0)}(1+w)}. \quad (41b)$$

These quantities are completely determined by the background equations (7) and the first-order equations (40). In the study of the evolution of density perturbations, it is convenient not to use $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ directly, but instead their corresponding contrast functions δ_ε and δ_n

$$\delta_\varepsilon(t, \mathbf{x}) \equiv \frac{\varepsilon_{(1)}^{\text{gi}}(t, \mathbf{x})}{\varepsilon_{(0)}(t)}, \quad \delta_n(t, \mathbf{x}) \equiv \frac{n_{(1)}^{\text{gi}}(t, \mathbf{x})}{n_{(0)}(t)}. \quad (42)$$

The system of equations (40) for the four independent quantities $\varepsilon_{(1)}$, $n_{(1)}$, $\vartheta_{(1)}$ and $R_{(1)}$ will now be rewritten, with the help of a computer algebra system, into a new system of equations for the two independent quantities δ_ε and δ_n . In this procedure it is explicitly assumed that $p \neq 0$, i.e., the pressure does not vanish identically. The case $p \rightarrow 0$ will be considered in Section III H on the non-relativistic limit. The final result is the generalised manifestly covariant

and gauge-invariant perturbation theory for the open, flat or closed FLRW universe

$$\ddot{\delta}_\varepsilon + b_1 \dot{\delta}_\varepsilon + b_2 \delta_\varepsilon = b_3 \left[\delta_n - \frac{\delta_\varepsilon}{1+w} \right], \quad (43a)$$

$$\frac{1}{c} \frac{d}{dt} \left[\delta_n - \frac{\delta_\varepsilon}{1+w} \right] = \frac{3Hn_{(0)}p_n}{\varepsilon_{(0)}(1+w)} \left[\delta_n - \frac{\delta_\varepsilon}{1+w} \right]. \quad (43b)$$

These are two linear differential equations for the two unknown quantities δ_ε and δ_n . It follows from $\beta^2 \equiv \dot{p}_{(0)}/\dot{\varepsilon}_{(0)}$ and equation (7c) that the time-derivative of $w \equiv p_{(0)}/\varepsilon_{(0)}$ is

$$\dot{w} = 3H(1+w)(w - \beta^2). \quad (44)$$

Defining $p_{nn} \equiv \partial^2 p / \partial n^2$ and $p_{\varepsilon n} \equiv \partial^2 p / \partial \varepsilon \partial n$ and using (44) the coefficients b_1 , b_2 and b_3 of equation (43a) are given by

$$b_1 = \frac{\kappa \varepsilon_{(0)}(1+w)}{H} - 2 \frac{\dot{\beta}}{\beta} - H(2+6w+3\beta^2) + R_{(0)} \left(\frac{1}{3H} + \frac{2H(1+3\beta^2)}{R_{(0)} + 3\kappa \varepsilon_{(0)}(1+w)} \right), \quad (45a)$$

$$b_2 = -\frac{1}{2} \kappa \varepsilon_{(0)}(1+w)(1+3w) + H^2(1-3w+6\beta^2(2+3w)) + 6H \frac{\dot{\beta}}{\beta} \left(w + \frac{\kappa \varepsilon_{(0)}(1+w)}{R_{(0)} + 3\kappa \varepsilon_{(0)}(1+w)} \right) \\ - R_{(0)} \left(\frac{1}{2}w + \frac{H^2(1+6w)(1+3\beta^2)}{R_{(0)} + 3\kappa \varepsilon_{(0)}(1+w)} \right) - \beta^2 \left(\frac{\tilde{\nabla}^2}{a^2} - \frac{1}{2}R_{(0)} \right), \quad (45b)$$

$$b_3 = \left\{ \frac{-18H^2}{R_{(0)} + 3\kappa \varepsilon_{(0)}(1+w)} \left[\varepsilon_{(0)}p_{\varepsilon n}(1+w) + \frac{2p_n}{3H} \frac{\dot{\beta}}{\beta} + p_n(p_\varepsilon - \beta^2) + n_{(0)}p_{nn} \right] + p_n \right\} \frac{n_{(0)}}{\varepsilon_{(0)}} \left(\frac{\tilde{\nabla}^2}{a^2} - \frac{1}{2}R_{(0)} \right). \quad (45c)$$

The equations (43) have been checked (see attached MAPLE file) using a computer algebra system, as follows. Substituting the contrast functions (42) in equations (43), where $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ are given by (41), and subsequently eliminating the time-derivatives of $\varepsilon_{(0)}$, $n_{(0)}$, H , $R_{(0)}$ and $\varepsilon_{(1)}$, $n_{(1)}$, $\vartheta_{(1)}$, $R_{(1)}$ with the help of equations (7) and (40), respectively, yields two identities for each of the two equations (43).

The equations (43) are two differential equations for the two independent and gauge-invariant quantities δ_ε and δ_n . It follows from equation (43b) that perturbations in the total energy density are *gravitationally* coupled to perturbations in the particle number density if $p_n \equiv (\partial p / \partial n)_\varepsilon \leq 0$, or, equivalently, $p_\varepsilon \equiv (\partial p / \partial \varepsilon)_n \geq \beta^2$, see (29). This is the case in a FLRW universe in the radiation-dominated era and in the era after decoupling of matter and radiation. This coupling is independent of the nature of the particles, i.e., it holds true for ordinary matter as well as CDM.

The system of equations (43) is equivalent to a system of *three* first-order differential equations, whereas the original set (40) is a *fourth-order* system. This difference is due to the fact that the gauge modes (38), which are solutions of the set (40), are completely removed from the solution set of (43): one degree of freedom, namely the gauge function ψ , has disappeared altogether.

The background equations (7) and the new perturbation equations (43) constitute a set of equations which enables one to study the evolution of small fluctuations in the energy density δ_ε and the particle number density δ_n in an open, flat or closed FLRW universe with $\Lambda \neq 0$ and filled with a perfect fluid with a non-barotropic equation of state for the pressure $p = p(n, \varepsilon)$.

F. Gauge-invariant Pressure and Temperature Perturbations

The gauge-invariant pressure and temperature perturbations, which are needed in the forthcoming sections, will now be derived.

From the equation of state (2) for the pressure $p = p(n, \varepsilon)$ it follows that

$$\dot{p}_{(0)} = p_n \dot{n}_{(0)} + p_\varepsilon \dot{\varepsilon}_{(0)}, \quad p_n \equiv \left(\frac{\partial p}{\partial n} \right)_\varepsilon, \quad p_\varepsilon \equiv \left(\frac{\partial p}{\partial \varepsilon} \right)_n. \quad (46)$$

Multiplying both sides of this expression by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from $p_{(1)}$ given by (11), one gets, using also (39a),

$$p_{(1)} - \frac{\dot{p}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)} = p_n n_{(1)}^{\text{gi}} + p_\varepsilon \varepsilon_{(1)}^{\text{gi}}. \quad (47)$$

Hence, the quantity defined by

$$p_{(1)}^{\text{gi}} \equiv p_{(1)} - \frac{\dot{p}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)}, \quad (48)$$

is the gauge-invariant pressure perturbation.

From the equation of state (1) for the energy density $\varepsilon = \varepsilon(n, T)$ it follows that

$$\dot{\varepsilon}_{(0)} = \left(\frac{\partial \varepsilon}{\partial n} \right)_T \dot{n}_{(0)} + \left(\frac{\partial \varepsilon}{\partial T} \right)_n \dot{T}_{(0)}, \quad \varepsilon_{(1)} = \left(\frac{\partial \varepsilon}{\partial n} \right)_T n_{(1)} + \left(\frac{\partial \varepsilon}{\partial T} \right)_n T_{(1)}. \quad (49)$$

Multiplying $\dot{\varepsilon}_{(0)}$ by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from $\varepsilon_{(1)}$, one finds, using (39a),

$$\varepsilon_{(1)}^{\text{gi}} = \left(\frac{\partial \varepsilon}{\partial n} \right)_T n_{(1)}^{\text{gi}} + \left(\frac{\partial \varepsilon}{\partial T} \right)_n \left[T_{(1)} - \frac{\dot{T}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)} \right], \quad (50)$$

implying that the quantity defined by

$$T_{(1)}^{\text{gi}} \equiv T_{(1)} - \frac{\dot{T}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)}, \quad (51)$$

is the gauge-invariant temperature perturbation. The expressions (48) and (51) are both of the form (39).

G. Diabatic Density Perturbations

In this section equations (43) will be linked to thermodynamics and it will be shown that, in general, density perturbations evolve diabatically, i.e., they exchange heat with their environment during their evolution.

The combined First and Second Law of Thermodynamics is given by

$$dE = TdS - pdV + \mu dN, \quad (52)$$

where E , S and N are the energy, the entropy and the number of particles of a system with volume V and pressure p , and where μ , the thermal—or chemical—potential, is the energy needed to add one particle to the system. In terms of the particle number density $n = N/V$, the energy per particle $E/N = \varepsilon/n$ and the entropy per particle $s = S/N$ the law (52) can be rewritten

$$d\left(\frac{\varepsilon}{n}N\right) = Td(sN) - pd\left(\frac{N}{n}\right) + \mu dN, \quad (53)$$

where ε is the energy density. The system is *extensive*, i.e., $S(\alpha E, \alpha V, \alpha N) = \alpha S(E, V, N)$, implying that the entropy of the gas is $S = (E + pV - \mu N)/T$. Dividing this relation by N one gets the so-called Euler relation

$$\mu = \frac{\varepsilon + p}{n} - Ts. \quad (54)$$

Eliminating μ in (53) with the help of (54), one finds that the combined First and Second Law of Thermodynamics (52) can be cast in a form without μ and N , i.e.,

$$Tds = d\left(\frac{\varepsilon}{n}\right) + pd\left(\frac{1}{n}\right). \quad (55)$$

From the background equations (7) it follows that $\dot{s}_{(0)} = 0$, implying with (37) that $s_{(1)} = s_{(1)}^{\text{gi}}$ is automatically gauge-invariant.

The thermodynamic relation (55) can, using (42), be rewritten in the form

$$T_{(0)} s_{(1)}^{\text{gi}} = -\frac{\varepsilon_{(0)}(1+w)}{n_{(0)}} \left[\delta_n - \frac{\delta_\varepsilon}{1+w} \right]. \quad (56)$$

Thus, the right-hand side of (43a) is related to local perturbations in the entropy.

Adiabatic perturbations do not exchange heat with their environment, so that $T_{(0)}s_{(1)}^{\text{gi}} = 0$. This implies with (56) that $(1+w)\delta_n - \delta_\varepsilon = 0$. Multiplying this expression by $3H\varepsilon_{(0)}n_{(0)}$ and substituting (42) one finds from the background conservation laws (7c) and (7e) that the adiabatic condition $s_{(1)}^{\text{gi}} = 0$ reads $\dot{n}_{(0)}\varepsilon_{(1)}^{\text{gi}} - \dot{\varepsilon}_{(0)}n_{(1)}^{\text{gi}} = 0$. Using that $\varepsilon = \varepsilon(n, T)$ the latter expression becomes

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_n \left[\dot{n}_{(0)}T_{(1)}^{\text{gi}} - n_{(1)}^{\text{gi}}\dot{T}_{(0)}\right] = 0. \quad (57)$$

Since n and T are independent quantities and since in a non-static universe one has $\dot{n}_{(0)} \neq 0$ and $\dot{T}_{(0)} \neq 0$, the adiabatic condition (57) is satisfied if, and only if,

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_n = 0, \quad (58)$$

implying that $\varepsilon = \varepsilon(n)$. In particular, in the non-relativistic limit, where $\varepsilon = nmc^2$ and $p = 0$, density perturbations are adiabatic. Hence, in all other cases where $p = p(n, \varepsilon)$ local density perturbations evolve *adiabatically*.

H. Non-relativistic Limit

In Section III D it has been shown that the two gauge-invariant quantities $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ are unique. It will be demonstrated that in the non-relativistic limit equations (28) combined with (39) reduce to the results (66) and (67) of the Newtonian Theory of Gravity and that the quantities $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ become equal to their Newtonian counterparts.

The non-relativistic limit is defined by three requirements, Carroll [17]. Firstly, the gravitational field should be weak, i.e., can be considered as a perturbation of flat space. Secondly, the particles are moving slowly with respect to the speed of light. Finally, the gravitational field of a density perturbation should be static, i.e., it does not change with time. This definition of the non-relativistic limit, which is essential to put an accurate interpretation on the quantities $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$, has not been used in former perturbation theories [5–10] to explain the meaning of the gauge-invariant variables.

In a first-order cosmological perturbation theory the gravitational field is already weak. In order to meet the first requirement, a flat ($R_{(0)} = 0$) FLRW universe is considered. Using (19), the local perturbation to the spatial curvature (15) reduces for a flat FLRW universe to

$$R_{(1)} = \frac{4}{c^2}\phi^{||k}{}_{|k} = -\frac{4}{c^2}\frac{\nabla^2\phi}{a^2}, \quad (59)$$

where ∇^2 is the usual Laplace operator. Substituting this expression into the perturbation equations (28), one gets

$$H(\theta_{(1)} - \vartheta_{(1)}) + \frac{1}{c^2}\frac{\nabla^2\phi}{a^2} = \frac{4\pi G_N}{c^4} \left[\varepsilon_{(1)}^{\text{gi}} + \frac{\dot{\varepsilon}_{(0)}}{\dot{\theta}_{(0)}}\theta_{(1)} \right], \quad (60a)$$

$$\frac{\nabla^2\dot{\phi}}{a^2} + \frac{4\pi G_N}{c^2}\varepsilon_{(0)}(1+w)\vartheta_{(1)} = 0, \quad (60b)$$

$$\dot{\varepsilon}_{(1)} + 3H(\varepsilon_{(1)} + p_{(1)}) + \varepsilon_{(0)}(1+w)\theta_{(1)} = 0, \quad (60c)$$

$$\dot{\vartheta}_{(1)} + H(2 - 3\beta^2)\vartheta_{(1)} + \frac{1}{\varepsilon_{(0)}(1+w)}\frac{\nabla^2 p_{(1)}}{a^2} = 0, \quad (60d)$$

$$\dot{n}_{(1)} + 3Hn_{(1)} + n_{(0)}\theta_{(1)} = 0, \quad (60e)$$

where (39a) has been used to eliminate $\varepsilon_{(1)}$ from the constraint equation (28a).

Next, the second requirement will be implemented. Since the spatial part $u_{(1)}^i$ of the fluid four-velocity is gauge dependent with a physical component and a non-physical gauge part, the second requirement should be defined by [18]

$$u_{(1)\text{ physical}}^i \equiv c^{-1}U_{(1)\text{ physical}}^i \rightarrow 0, \quad (61)$$

i.e., the *physical* part of the spatial part of the fluid four-velocity is negligible with respect to the speed of light. In this limit, the mean kinetic energy per particle $\frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}k_B T \rightarrow 0$ is very small compared to the rest energy mc^2 per particle. This implies that the pressure $p = nk_B T \rightarrow 0$ ($n \neq 0$) is vanishingly small with respect to the rest

energy density nmc^2 . Substituting $p = 0$ into the momentum conservation law (10e) yields, using also the background equation (7c) with $w \equiv p_{(0)}/\varepsilon_{(0)} \rightarrow 0$,

$$\dot{u}_{(1)}^i = -2Hu_{(1)}^i. \quad (62)$$

Since the physical part of $u_{(1)}^i$ vanishes in the non-relativistic limit, the general solution of equation (62) is exactly equal to the gauge mode (38b)

$$\hat{u}_{(1)}^i(t, \mathbf{x}) = -\frac{1}{a^2(t)} \tilde{g}^{ik}(\mathbf{x}) \partial_k \psi(\mathbf{x}), \quad (63)$$

where it is used that $H \equiv \dot{a}/a$. Thus, in the limit (61) one is left with the gauge mode (63) only. Consequently, one may, without losing any physical information, put the gauge mode $\hat{u}_{(1)}^i$ equal to zero, implying that $\partial_k \psi = 0$, so that ψ is a constant in the non-relativistic limit. Substituting $\partial_k \psi = 0$ into (5) one finds that the relativistic transformation (3) between synchronous coordinates reduces in the limit (61) to the (infinitesimal) transformation

$$x^0 \rightarrow x^0 - \psi, \quad x^i \rightarrow x^i - \chi^i(\mathbf{x}), \quad (64)$$

where ψ is an arbitrary constant and $\chi^i(\mathbf{x})$ are three arbitrary functions of the spatial coordinates. In the non-relativistic limit time and space transformation are decoupled: time coordinates may be shifted and spatial coordinates may be chosen arbitrarily. The residual gauge freedom ψ and $\chi^i(\mathbf{x})$ may not come as a surprise, since the Newtonian Theory of Gravity is invariant under the gauge transformation (64).

Substituting $\vartheta_{(1)} = 0$ and $p = 0$ into the system (60), one gets

$$\nabla^2 \phi = \frac{4\pi G_N}{c^2} a^2 \varepsilon_{(1)}^{\text{gi}}, \quad (65a)$$

$$\nabla^2 \dot{\phi} = 0, \quad (65b)$$

$$\dot{\varepsilon}_{(1)} + 3H\varepsilon_{(1)} + \varepsilon_{(0)}\theta_{(1)} = 0, \quad (65c)$$

$$\dot{n}_{(1)} + 3Hn_{(1)} + n_{(0)}\theta_{(1)} = 0. \quad (65d)$$

The constraint equation (65a) can be found by subtracting $\frac{1}{6}\theta_{(1)}/\dot{H}$ times the time-derivative of the background constraint equation (7a) with $R_{(0)} = 0$ from the constraint equation (60a) and using that $\theta_{(0)} = 3H$. Note that the cosmological constant Λ need not be zero.

Since $\vartheta_{(1)} = 0$, i.e., no fluid flow, density perturbations do not evolve, so that their gravitational field is *static* (65b). Consequently, $a^2(t)\varepsilon_{(1)}^{\text{gi}}(t, \mathbf{x})$ in (65a) should be replaced by $a^2(t_0)\varepsilon_{(1)}^{\text{gi}}(t_0, \mathbf{x})$. Defining the potential $\varphi(\mathbf{x}) \equiv \phi(\mathbf{x})/a^2(t_0)$, equations (65a) and (65b) imply

$$\nabla^2 \varphi(\mathbf{x}) = 4\pi G_N \rho_{(1)}(\mathbf{x}), \quad \rho_{(1)}(\mathbf{x}) \equiv \frac{\varepsilon_{(1)}^{\text{gi}}(t_0, \mathbf{x})}{c^2}, \quad (66)$$

which is the Poisson equation of the Newtonian Theory of Gravity. With (66) the third requirement for the non-relativistic limit, i.e., a static gravitational field, has been satisfied. The universe is in the non-relativistic limit not static, since $H \neq 0$ and $\dot{H} \neq 0$, as follows from the background equations (7) with $w = 0$ and $R_{(0)} = 0$. In the non-relativistic limit a local density perturbation does not follow the global expansion of the universe.

Equations (65c) and (65d) have no physical significance since the only solutions of these equations are the gauge modes (38a) with constant ψ , (64). Therefore, the latter two equations are not part of the Newtonian Theory of Gravity and need not be considered. The potential ζ which occurs by (19) in $R_{(1)}$, (15), and $\theta_{(1)}$, (16), in the general relativistic case, drops from the perturbation theory in the non-relativistic limit. Consequently, one is left with one potential $\varphi(\mathbf{x})$ only.

The expression (41a) reduces in the non-relativistic limit to $\varepsilon_{(1)}^{\text{gi}} = -R_{(1)}/(2\kappa)$, which is, with (59) and (65b), equivalent to the Poisson equation (66). Expression (41b) reduces in the non-relativistic limit to the familiar result

$$n_{(1)}^{\text{gi}} = \frac{\varepsilon_{(1)}^{\text{gi}}}{mc^2}, \quad (67)$$

where it is used that, in the non-relativistic limit, $\varepsilon_{(1)} = n_{(1)}mc^2$ and $\varepsilon_{(0)} = n_{(0)}mc^2$.

It has been shown that equations (28) combined with the *unique* gauge-invariant quantities (39) reduces in the non-relativistic limit to the Newtonian results (66) and (67). Consequently, $\varepsilon_{(1)}^{\text{gi}}$ and $n_{(1)}^{\text{gi}}$ are the real, physical perturbations to the energy density and particle number density, respectively.

IV. EXAMPLE: THE FLAT FLRW UNIVERSE

In this section analytic solutions of equations (43) are derived for a flat ($R_{(0)} = 0$) FLRW universe with a vanishing cosmological constant ($\Lambda = 0$) in its radiation-dominated phase and in the era after decoupling of matter and radiation.

A. Radiation-dominated Era

In the radiation-dominated epoch one has $\varepsilon = a_B T_\gamma^4$, where a_B is the black body constant and T_γ the radiation temperature. The pressure is $p = \frac{1}{3}\varepsilon$, so that $p_n = 0$, $p_\varepsilon = \frac{1}{3}$, implying, with (29), that $\beta^2 = \frac{1}{3}$. The perturbation equations (43) reduce to

$$\ddot{\delta}_\varepsilon - H\dot{\delta}_\varepsilon - \left[\frac{1}{3} \frac{\nabla^2}{a^2} - \frac{2}{3} \kappa \varepsilon_{(0)} \right] \delta_\varepsilon = 0, \quad (68a)$$

$$\frac{1}{c} \frac{d}{dt} (\delta_n - \frac{3}{4} \delta_\varepsilon) = 0. \quad (68b)$$

Equation (68b) implies that the difference $\delta_n - \frac{3}{4} \delta_\varepsilon$ depends only on the spatial coordinates. Since (68a) has no source term, heat exchange is unimportant in the radiation-dominated era. Therefore, one may use the adiabatic condition $\delta_n = \frac{3}{4} \delta_\varepsilon$, see Section III G.

Equation (68a) will now be rewritten in dimensionless quantities. The solutions of the background equations (7) are given by

$$H \propto t^{-1}, \quad \varepsilon_{(0)} \propto t^{-2}, \quad n_{(0)} \propto t^{-3/2}, \quad a \propto t^{1/2}, \quad (69)$$

implying that $T_{(0)\gamma} \propto a^{-1}$. The dimensionless time τ is defined by $\tau \equiv t/t_0$. Since $H \equiv \dot{a}/a$, one finds that

$$\frac{d^k}{c^k dt^k} = \left[\frac{1}{ct_0} \right]^k \frac{d^k}{d\tau^k} = [2H(t_0)]^k \frac{d^k}{d\tau^k}, \quad k = 1, 2. \quad (70)$$

Substituting $\delta_\varepsilon(t, \mathbf{x}) = \delta_\varepsilon(t, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x})$ into equation (68a) and using (70) yields

$$\delta_\varepsilon'' - \frac{1}{2\tau} \delta_\varepsilon' + \left[\frac{\mu_r^2}{4\tau} + \frac{1}{2\tau^2} \right] \delta_\varepsilon = 0, \quad \tau \geq 1, \quad (71)$$

where a prime denotes differentiation with respect to τ . The parameter μ_r is given by

$$\mu_r \equiv \frac{2\pi}{\lambda_0} \frac{1}{H(t_0)} \frac{1}{\sqrt{3}}, \quad \lambda_0 \equiv \lambda a(t_0), \quad (72)$$

with $\lambda a(t_0)$ the physical scale of a perturbation at time t_0 , and $|\mathbf{q}| = 2\pi/\lambda$. To solve equation (71), replace τ by $x \equiv \mu_r \sqrt{\tau}$. After transforming back to τ , one finds

$$\delta_\varepsilon(\tau, \mathbf{q}) = \left[A_1(\mathbf{q}) \sin(\mu_r \sqrt{\tau}) + A_2(\mathbf{q}) \cos(\mu_r \sqrt{\tau}) \right] \sqrt{\tau}, \quad (73)$$

where the ‘constants’ of integration $A_1(\mathbf{q})$ and $A_2(\mathbf{q})$ are given by

$$A_1(\mathbf{q}) = \delta_\varepsilon(t_0, \mathbf{q}) \sin \mu_r - \frac{\cos \mu_r}{\mu_r} \left[\delta_\varepsilon(t_0, \mathbf{q}) - \frac{\dot{\delta}_\varepsilon(t_0, \mathbf{q})}{H(t_0)} \right], \quad (74a)$$

$$A_2(\mathbf{q}) = \delta_\varepsilon(t_0, \mathbf{q}) \cos \mu_r + \frac{\sin \mu_r}{\mu_r} \left[\delta_\varepsilon(t_0, \mathbf{q}) - \frac{\dot{\delta}_\varepsilon(t_0, \mathbf{q})}{H(t_0)} \right]. \quad (74b)$$

For large-scale perturbations ($\lambda \rightarrow \infty$), it follows from (73) and (74) that

$$\delta_\varepsilon(t) = - \left[\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \frac{t}{t_0} + \left[2\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \left(\frac{t}{t_0} \right)^{\frac{1}{2}}. \quad (75)$$

The energy density contrast has two contributions to the growth rate, one proportional to t and one proportional to $t^{1/2}$. These two solutions have been found, with the exception of the precise factors of proportionality, by a large number of authors, see Lifshitz and Khalatnikov [3], (8.11), Adams and Canuto [19], (4.5b), Olson [20], page 329, Peebles [21], (86.20), Kolb and Turner [22], (9.121) and Press and Vishniac [23], (33). Consequently, the generalised Lifshitz-Khalatnikov theory corroborates for large-scale perturbations the results of the literature.

A new result is that small-scale perturbations ($\lambda \rightarrow 0$) oscillate with an *increasing* amplitude according to

$$\delta_\varepsilon(t, \mathbf{q}) \approx \delta_\varepsilon(t_0, \mathbf{q}) \left(\frac{t}{t_0} \right)^{\frac{1}{2}} \cos \left[\mu_r - \mu_r \left(\frac{t}{t_0} \right)^{\frac{1}{2}} \right], \quad (76)$$

as follows from (73) and (74). These small-scale perturbations manifest themselves as small temperature fluctuations in the cosmic background radiation.

By virtue of equation (68b) particle number density fluctuations δ_n are coupled to fluctuations δ_ε in the energy density. Since equation (68b) is independent of the nature of the particles and CDM interacts only via gravity with ordinary matter and radiation, the fluctuations in CDM are gravitationally coupled to fluctuations in the energy density. In other words, $\delta_n = \frac{3}{4}\delta_\varepsilon$ holds true for ordinary matter as well as CDM. Consequently, in the radiation-dominated universe CDM does not contract faster than ordinary matter, so that star formation can only commence after decoupling.

The standard equation (104) with $w = \frac{1}{3}$ predicts oscillating density perturbations with a *constant* amplitude. As will be explained in Section VI, this outcome is to be expected since in the standard equation the divergence of the spatial part of the fluid four-velocity, $\vartheta_{(1)}$, has not been taken into account. If $\vartheta_{(1)} = 0$ there is no fluid flow, implying that a density perturbation does not grow. In the new theory (68) $\vartheta_{(1)} \neq 0$, so that oscillating density perturbations with an increasing amplitude (76) are found.

B. Era after Decoupling of Matter and Radiation

Once protons and electrons combine to yield hydrogen, the radiation pressure becomes negligible, and the equations of state (1) become those of a non-relativistic monatomic perfect gas

$$\varepsilon(n, T) = nmc^2 + \frac{3}{2}nk_B T, \quad p(n, T) = nk_B T, \quad (77)$$

where k_B is Boltzmann's constant, m the mean particle mass, and T the temperature of the matter. It is assumed that the CDM particle mass is larger than or equal to the proton mass, $m_{\text{CDM}} \geq m_H$, implying that for the mean particle mass m one has $mc^2 \gg k_B T$, so that $w \equiv p_{(0)}/\varepsilon_{(0)} \ll 1$. Therefore, as follows from the background equations (7a) and (7c), one may neglect the pressure $nk_B T$ and the kinetic energy density $\frac{3}{2}nk_B T$ with respect to the rest-mass energy density nmc^2 in the *unperturbed* universe. However, neglecting the pressure in the perturbed universe yields density perturbations with a *static* gravitational field as has been demonstrated in Section III H.

Eliminating T from (77) yields $p(n, \varepsilon) = \frac{2}{3}(\varepsilon - nmc^2)$, so that $p_\varepsilon \equiv (\partial p / \partial \varepsilon)_n = \frac{2}{3}$ and $p_n \equiv (\partial p / \partial n)_\varepsilon = -\frac{2}{3}mc^2$. Substituting p_n , p_ε and (77) into (29) on finds, using $mc^2 \gg k_B T$,

$$\beta(t) \approx \frac{v_s(t)}{c} = \sqrt{\frac{5}{3} \frac{k_B T_{(0)}(t)}{mc^2}}, \quad T_{(0)} \propto a^{-2}, \quad (78)$$

with v_s the adiabatic speed of sound and $T_{(0)}$ the matter temperature. The fact that $T_{(0)} \propto a^{-2}$ follows from the equations of state (77) and the conservation laws (7c) and (7e). This, in turn, implies with (78) that $\dot{\beta}/\beta = -H$. The system (43) can now be rewritten as

$$\ddot{\delta}_\varepsilon + 3H\dot{\delta}_\varepsilon - \left[\beta^2 \frac{\nabla^2}{a^2} + \frac{5}{6}\kappa\varepsilon_{(0)} \right] \delta_\varepsilon = -\frac{2}{3} \frac{\nabla^2}{a^2} (\delta_n - \delta_\varepsilon), \quad (79a)$$

$$\frac{1}{c} \frac{d}{dt} (\delta_n - \delta_\varepsilon) = -2H (\delta_n - \delta_\varepsilon), \quad (79b)$$

where $w \ll 1$ and $\beta^2 \ll 1$ have been neglected with respect to constants of order unity.

From equation (79b) it follows that

$$\delta_n - \delta_\varepsilon \propto a^{-2}, \quad (80)$$

where it is used that $H \equiv \dot{a}/a$.

Using that $k_B T_{(0)} \ll mc^2$, one finds for the perturbed counterparts of (77)

$$\delta_n - \delta_\varepsilon \approx -\frac{3}{2} \frac{k_B T_{(0)}}{mc^2} \delta_T, \quad \delta_p = \delta_n + \delta_T, \quad (81)$$

where δ_p is the relative pressure perturbation defined by $\delta_p \equiv p_{(1)}^{\text{gi}}/p_{(0)}$ and δ_T is the relative matter temperature perturbation defined by $\delta_T \equiv T_{(1)}^{\text{gi}}/T_{(0)}$, see (48) and (51). Combining (78) and (80) one finds from (81) that δ_T is nearly constant, i.e.,

$$\delta_T(t, \mathbf{x}) \approx \delta_T(t_0, \mathbf{x}), \quad (82)$$

to a very good approximation.

From (78) and (81) it follows that the source term of equation (79a) is of the same order of magnitude as the term with coefficient β^2 . It follows from (45c) that the source term vanishes for barotropic equations of state $p = p(\varepsilon)$. As will become clear in Section V, the source term of equation (79a) is crucial for the understanding of star formation in the early universe after decoupling. That is why the realistic equation of state $p = p(n, \varepsilon)$ has been incorporated from the outset in the perturbation theory (43).

Equation (79a) will now be rewritten in dimensionless quantities. The solutions of the background equations (7) are given by

$$H \propto t^{-1}, \quad \varepsilon_{(0)} \propto t^{-2}, \quad n_{(0)} \propto t^{-2}, \quad a \propto t^{2/3}, \quad (83)$$

where the kinetic energy density and pressure have been neglected with respect to the rest-mass energy density. The dimensionless time τ is defined by $\tau \equiv t/t_0$. Using that $H \equiv \dot{a}/a$, one gets

$$\frac{d^k}{c^k dt^k} = \left[\frac{1}{ct_0} \right]^k \frac{d^k}{d\tau^k} = \left[\frac{3}{2} H(t_0) \right]^k \frac{d^k}{d\tau^k}, \quad k = 1, 2. \quad (84)$$

Substituting $\delta_\varepsilon(t, \mathbf{x}) = \delta_\varepsilon(t, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x})$ and $\delta_n(t, \mathbf{x}) = \delta_n(t, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x})$ into equations (79) and using (81) and (84) one finds that equations (79) can be combined into one equation

$$\delta_\varepsilon'' + \frac{2}{\tau} \delta_\varepsilon' + \left[\frac{4}{9} \frac{\mu_m^2}{\tau^{8/3}} - \frac{10}{9\tau^2} \right] \delta_\varepsilon = -\frac{4}{15} \frac{\mu_m^2}{\tau^{8/3}} \delta_T(t_0, \mathbf{q}), \quad \tau \geq 1, \quad (85)$$

where a prime denotes differentiation with respect to τ . The parameter μ_m is given by

$$\mu_m \equiv \frac{2\pi}{\lambda_0} \frac{1}{H(t_0)} \frac{v_s(t_0)}{c}, \quad \lambda_0 \equiv \lambda a(t_0), \quad (86)$$

with $\lambda a(t_0)$ the physical scale of a perturbation at time t_0 , and $|\mathbf{q}| = 2\pi/\lambda$. To solve equation (85) replace τ by $\tau \equiv 2\mu_m \tau^{-1/3}$. After transforming back to τ , one finds for the general solution of (85)

$$\delta_\varepsilon(\tau, \mathbf{q}) = \left[B_1(\mathbf{q}) J_{+\frac{7}{2}}(2\mu_m \tau^{-1/3}) + B_2(\mathbf{q}) J_{-\frac{7}{2}}(2\mu_m \tau^{-1/3}) \right] \tau^{-1/2} - \frac{3}{5} \left[1 + \frac{5\tau^{2/3}}{2\mu_m^2} \right] \delta_T(t_0, \mathbf{q}), \quad (87)$$

where $J_{\pm 7/2}(x)$ are Bessel functions of the first kind and $B_1(\mathbf{q})$ and $B_2(\mathbf{q})$ are the ‘constants’ of integration.

In the large-scale limit $\lambda \rightarrow \infty$ terms with ∇^2 vanish, so that the general solution of equation (85) is

$$\delta_\varepsilon(t) = \frac{1}{7} \left[5\delta_\varepsilon(t_0) + \frac{2\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \left(\frac{t}{t_0} \right)^{\frac{2}{3}} + \frac{2}{7} \left[\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \left(\frac{t}{t_0} \right)^{-\frac{5}{3}}. \quad (88)$$

Thus, for large-scale perturbations the initial value $\delta_T(t_0, \mathbf{q})$ does not play a role during the evolution: large-scale perturbations evolve only under the influence of gravity. These perturbations are so large that heat exchange does not play a role during their evolution in the linear phase. For perturbations much larger than the Jeans scale (i.e., the peak value in Figure 1), gravity alone is insufficient to explain star formation within 13.75 Gyr, since they grow as $\delta_\varepsilon \propto t^{2/3}$. The solution proportional to $t^{2/3}$ is a standard result. Since δ_ε is gauge-invariant, the standard non-physical gauge mode proportional to t^{-1} is absent from the new theory. Instead, a physical mode proportional to $t^{-5/3}$ is found. This mode has also been found by Bardeen [5], Table I, and by Mukhanov *et al.* [9], expression (5.33). In order

to arrive at the $t^{-5/3}$ mode, Bardeen has to use the ‘uniform expansion gauge.’ In the generalised Lifshitz-Khalatnikov theory the Hubble function is *automatically* uniform, (39b), without any additional gauge condition. Consequently, (88) is in agreement with results given in the literature.

In the small-scale limit $\lambda \rightarrow 0$, one finds

$$\delta_\varepsilon(t, \mathbf{q}) \approx -\frac{3}{5}\delta_T(t_0, \mathbf{q}) + \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \left[\frac{3}{5}\delta_T(t_0, \mathbf{q}) + \delta_\varepsilon(t_0, \mathbf{q}) \right] \cos \left[2\mu_m - 2\mu_m \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \right]. \quad (89)$$

Thus, density perturbations with scales much smaller than the Jeans scale oscillate with a decaying amplitude which is smaller than unity: these perturbations are so small that gravity is insufficient to let perturbations grow. Heat loss alone is not enough for the growth of density perturbations. Consequently, perturbations with scales much smaller than the Jeans scale will never reach the non-linear regime.

In the next section it is shown that for density perturbations with scales of the order of the Jeans scale, the action of both gravity and heat loss together may result in massive stars several hundred million years after decoupling of matter and radiation.

V. EVOLUTION OF SMALL-SCALE INHOMOGENEITIES AFTER DECOUPLING

In this section it is demonstrated that the generalised Lifshitz-Khalatnikov perturbation theory based on the General Theory of Relativity combined with thermodynamics and a realistic equation of state for the pressure $p = p(n, \varepsilon)$, predicts that in the era after decoupling of matter and radiation small-scale inhomogeneities may grow very fast. As in Section IV, a flat ($R_{(0)} = 0$) FLRW universe with vanishing cosmological constant ($\Lambda = 0$) is considered.

A. Introducing Observable Quantities

The parameter μ_m (86) will be expressed in observable quantities, namely the present values of the background radiation temperature, $T_{(0)\gamma}(t_p)$, the Hubble parameter, $\mathcal{H}(t_p) = cH(t_p)$, and the redshift at decoupling, $z(t_{\text{dec}})$.

The redshift $z(t)$ as a function of the scale factor $a(t)$ is given by

$$z(t) = \frac{a(t_p)}{a(t)} - 1, \quad (90)$$

where $a(t_p)$ is the present value of the scale factor. For a flat FLRW universe one may take $a(t_p) = 1$.

Substituting (78) into (86), one gets

$$\mu_m = \frac{2\pi}{\lambda_{\text{dec}}} \frac{1}{H(t_{\text{dec}})} \sqrt{\frac{5}{3} \frac{k_B T_{(0)}(t_{\text{dec}})}{mc^2}}, \quad \lambda_{\text{dec}} \equiv \lambda a(t_{\text{dec}}), \quad (91)$$

where t_{dec} is the time when a perturbation starts to contract and λ_{dec} the physical scale of a perturbation at time t_{dec} . Using (83) and (90), one finds

$$\mu_m = \frac{2\pi}{\lambda_{\text{dec}}} \frac{1}{\mathcal{H}(t_p)[z(t_{\text{dec}}) + 1]} \sqrt{\frac{5}{3} \frac{k_B T_{(0)\gamma}(t_p)}{m}}, \quad \lambda_{\text{dec}} \equiv \lambda a(t_{\text{dec}}), \quad (92)$$

where it is used that $T_{(0)}(t_{\text{dec}}) = T_{(0)\gamma}(t_{\text{dec}})$, and that $T_{(0)\gamma} \propto a^{-1}$ after decoupling, as follows from (69). With (92) the parameter μ_m is expressed in observable quantities.

B. Initial Values from WMAP

The physical quantities measured by the Wilkinson Microwave Anisotropy Probe (WMAP) [24] and needed in the parameter μ_m (92) of the generalised Lifshitz-Khalatnikov theory is the redshift at decoupling, the present values of the Hubble function and the background radiation temperature, the age of the universe and the fluctuations in the

background radiation temperature. The numerical values of these quantities are

$$z(t_{\text{dec}}) = 1091, \quad (93a)$$

$$cH(t_p) = \mathcal{H}(t_p) = 71.0 \text{ km/sec/Mpc} = 2.30 \times 10^{-18} \text{ sec}^{-1}, \quad (93b)$$

$$T_{(0)\gamma}(t_p) = 2.725 \text{ K}, \quad (93c)$$

$$t_p = 13.75 \text{ Gyr}, \quad (93d)$$

$$\delta_{T_\gamma}(t_{\text{dec}}) \lesssim 10^{-5}. \quad (93e)$$

Substituting the observed values (93a)–(93c) into (92), one finds

$$\mu_m = \frac{15.69}{\lambda_{\text{dec}}}, \quad \lambda_{\text{dec}} \text{ in pc}, \quad (94)$$

where it is used that $m = m_H = 1.6726 \times 10^{-27} \text{ kg}$ and $1 \text{ pc} = 3.0857 \times 10^{16} \text{ m} = 3.2616 \text{ ly}$. Boltzmann's constant is $k_B = 1.3806 \times 10^{-23} \text{ J K}^{-1}$.

The WMAP observations of the fluctuations $\delta_{T_\gamma}(t_{\text{dec}})$, (93e), in the background radiation temperature yield for the fluctuations in the energy density

$$|\delta_\varepsilon(t_{\text{dec}}, \mathbf{q})| \lesssim 10^{-5}. \quad (95)$$

In addition, it is assumed that

$$\dot{\delta}_\varepsilon(t_{\text{dec}}, \mathbf{q}) \approx 0, \quad (96)$$

i.e., during the transition from the radiation-dominated era to the era after decoupling, perturbations in the energy density are approximately constant with respect to time. It follows from (81) that during the linear phase of the evolution one has $\delta_n(t, \mathbf{q}) \approx \delta_\varepsilon(t, \mathbf{q})$, so that the initial values $\delta_n(t_{\text{dec}}, \mathbf{q})$ and $\dot{\delta}_n(t_{\text{dec}}, \mathbf{q})$ need not be specified.

C. Star Formation in the Early Universe

At the moment of decoupling of matter and radiation, photons could not ionise matter any more and the two constituents fell out of thermal equilibrium. As a consequence, the pressure drops from a very high radiation pressure $p = \frac{1}{3}a_B T_\gamma^4$ just before decoupling to a very low gas pressure $p = nk_B T$ after decoupling. This fast and chaotic transition from a high pressure epoch to a very low pressure era may result in large relative pressure perturbations $\delta_p \equiv p_{(1)}^{\text{gi}}/p_{(0)}$. With (81) and (95) it follows that $\delta_T \equiv T_{(1)}^{\text{gi}}/T_{(0)}$ could be large. As will be shown, relative initial pressure perturbations

$$\delta_p(t_{\text{dec}}, \mathbf{q}) \approx \delta_T(t_{\text{dec}}, \mathbf{q}) \lesssim -0.005, \quad (97)$$

may result in primordial stars, the so-called (hypothetical) population III stars, several hundred million years after the Big Bang.

The evolution equation (85) is solved numerically and the results are summarised in Figure 1, which is constructed as follows. For each choice of $\delta_T(t_{\text{dec}}, \mathbf{q})$ equation (85) is integrated for a large number of values for the initial perturbation scale λ_{dec} using the initial values (95) and (96). The integration starts at $\tau \equiv t/t_{\text{dec}} = 1$, i.e., at $z(t_{\text{dec}}) = 1091$ and will be halted if either $z = 0$ (i.e., $\tau = [z(t_{\text{dec}}) + 1]^{3/2}$), or $\delta_\varepsilon(t, \mathbf{q}) = 1$ for $z > 0$ has been reached. One integration run yields one point on the curve for a particular choice of the scale λ_{dec} if $\delta_\varepsilon(t, \mathbf{q}) = 1$ has been reached for $z > 0$. If the integration halts at $z(t_p) = 0$ and still $\delta_\varepsilon(t_p, \mathbf{q}) < 1$, then the perturbation belonging to that particular scale λ_{dec} has not yet reached its non-linear phase today, i.e., at $t_p = 13.75 \text{ Gyr}$. On the other hand, if the integration is stopped at $\delta_\varepsilon(t, \mathbf{q}) = 1$ and $z > 0$, then the perturbation has become non-linear within 13.75 Gyr. This procedure has been performed for $\delta_T(t_{\text{dec}}, \mathbf{q})$ in the range $-0.005, -0.01, -0.02, \dots, -0.1$. Each curve denotes the time and scale for which $\delta_\varepsilon(t, \mathbf{q}) = 1$ for a particular $\delta_T(t_{\text{dec}}, \mathbf{q})$.

The growth of a perturbation is governed by both gravity as well as heat loss. The smaller the scale of a density perturbation, the more it benefits from heat loss and the less from its internal gravity. From Figure 1 one may infer that the optimal scale for growth is around $6.2 \text{ pc} \approx 20 \text{ ly}$. At this scale, which is independent of the initial value of the matter temperature perturbation $\delta_T(t_{\text{dec}}, \mathbf{q})$, heat loss and gravity work together perfectly, resulting in a fast growth. Perturbations with scales smaller than 6.2 pc reach their non-linear phase at a later time, because their internal gravity is weaker than for large-scale perturbations. On the other hand, perturbations with scales larger than 6.2 pc cool down slower because of their large scales, resulting also in a smaller growth rate. Since the growth rate decreases rapidly for perturbations with scales below 6.2 pc , this scale will be considered as the *relativistic* counterpart of the classical *Jeans scale*. The relativistic Jeans scale is much smaller than the horizon size at decoupling, given by $d_H(t_{\text{dec}}) = 3ct_{\text{dec}} \approx 3.5 \times 10^5 \text{ pc} \approx 1.1 \times 10^6 \text{ ly}$.

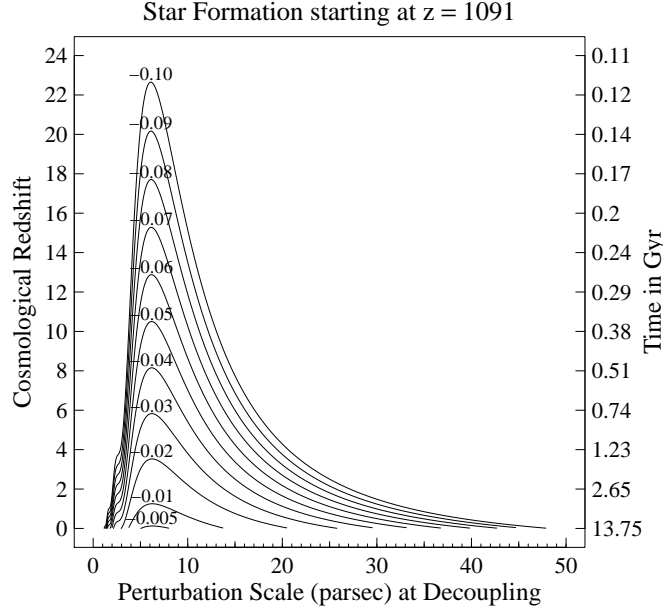


Figure 1. The curves give the redshift and time when a linear perturbation in the energy density with initial values $\delta_\varepsilon(t_{\text{dec}}, \mathbf{q}) \lesssim 10^{-5}$ and $\dot{\delta}_\varepsilon(t_{\text{dec}}, \mathbf{q}) \approx 0$ starting to grow at an initial redshift of $z(t_{\text{dec}}) = 1091$ becomes non-linear, i.e., $\delta_\varepsilon(t, \mathbf{q}) = 1$. The numbers at each of the curves are the initial relative perturbations $\delta_T(t_{\text{dec}}, \mathbf{q})$ in the matter temperature. For each curve, the Jeans scale (i.e., the peak value) is at 6.2 pc.

D. Heat Loss of a Density Perturbation during its Contraction

The heat loss of a density perturbation during its contraction after decoupling can be calculated from the combined First and Second Law of Thermodynamics (56) rewritten in the form

$$T_{(0)} s_{(1)}^{\text{gi}} = -\frac{\varepsilon_{(0)}}{n_{(0)}} (\delta_n - \delta_\varepsilon) - \frac{p_{(0)}}{n_{(0)}} \delta_n, \quad (98)$$

where it is used that $w \equiv p_{(0)}/\varepsilon_{(0)}$. Substituting expressions (77) and (81) into (98) and using also (82), one finds the entropy per particle of a density perturbation:

$$s_{(1)}^{\text{gi}}(t, \mathbf{x}) \approx \frac{1}{2} k_B [3\delta_T(t_0, \mathbf{x}) - 2\delta_n(t, \mathbf{x})], \quad (99)$$

where it is used that $mc^2 \gg k_B T_{(0)}$. From (99) it follows that for $\delta_T \leq 0$ and $\delta_n > 0$ the entropy perturbation is negative, $s_{(1)}^{\text{gi}} < 0$. Since for growing perturbations one has $\dot{\delta}_n > 0$ the entropy perturbation decreases, i.e., $\dot{s}_{(1)}^{\text{gi}} \approx -k_B \dot{\delta}_n < 0$, during contraction. This is to be expected, since a local density perturbation is not isolated from its environment. Only for an isolated system the entropy never decreases.

E. Relativistic Jeans Mass

The Jeans mass at decoupling, $M_J(t_{\text{dec}})$, can be estimated by assuming that a density perturbation has a spherical symmetry with diameter the relativistic Jeans scale $\lambda_J(t_{\text{dec}})$. The relativistic Jeans mass at decoupling is then given by

$$M_J(t_{\text{dec}}) = \frac{4\pi}{3} \left(\frac{1}{2}\lambda_{\text{dec}}\right)^3 n_{(0)}(t_{\text{dec}}) m. \quad (100)$$

The particle number density $n_{(0)}(t_{\text{dec}})$ can be calculated from its value $n_{(0)}(t_{\text{eq}})$ at the end of the radiation-dominated era. By definition, at the end of the radiation-domination era the matter energy density $n_{(0)}mc^2$ equals the energy density of the radiation:

$$n_{(0)}(t_{\text{eq}})mc^2 = a_B T_{(0)\gamma}^4(t_{\text{eq}}). \quad (101)$$

Since $n_{(0)} \propto a^{-3}$ and $T_{(0)\gamma} \propto a^{-1}$, one finds, using (90) and (101), for the Jeans mass (100) at time t_{dec}

$$M_J(t_{\text{dec}}) = \frac{1}{6} \pi \lambda_{\text{dec}}^3 \frac{a_B T_{(0)\gamma}^4(t_p)}{c^2} [z(t_{\text{eq}}) + 1] [z(t_{\text{dec}}) + 1]^3. \quad (102)$$

Using (93a), the black body constant $a_B = 7.5657 \times 10^{-16} \text{ J/m}^3/\text{K}^4$, the red-shift at matter-radiation equality, $z(t_{\text{eq}}) = 3196$, WMAP [24], and the speed of light $c = 2.9979 \times 10^8 \text{ m/s}$, one finds for the Jeans mass at decoupling

$$M_J(t_{\text{dec}}) \approx 3.5 \times 10^3 M_\odot, \quad (103)$$

where it is used that one solar mass $1 M_\odot = 1.9889 \times 10^{30} \text{ kg}$ and the relativistic Jeans scale $\lambda_J(t_{\text{dec}}) = 6.2 \text{ pc}$, the peak value in Figure 1.

VI. WHY THE STANDARD EVOLUTION EQUATION IS NO LONGER ADEQUATE

The standard evolution equation for relative density perturbations $\delta(t, \mathbf{x})$ in a flat, $R_{(0)} = 0$, FLRW universe with vanishing cosmological constant, $\Lambda = 0$, reads

$$\ddot{\delta} + 2H\dot{\delta} - \left[\frac{v_s^2}{c^2} \frac{\nabla^2}{a^2} + \frac{1}{2} \kappa \varepsilon_{(0)} (1+w)(1+3w) \right] \delta = 0. \quad (104)$$

As will become clear in the sequel, this equation holds true if and only if the pressure $p_{(0)}$ and the energy density $\varepsilon_{(0)}$ and their perturbations $p_{(1)}$ and $\varepsilon_{(1)}$ obey the linear barotropic equation of state

$$p = w\varepsilon. \quad (105)$$

The evolution equation (104) is derived from the Newtonian Theory of Gravity. In this derivation it is assumed that $(v_s/c)^2 = w = \frac{1}{3}$ in the radiation-dominated era, and $w = 0$ and v_s/c is time dependent and given by (78) after decoupling of matter and radiation.

The fact that the factor of proportionality w is a constant can be demonstrated using the general expressions (29) and (44). From (11) and (105) one finds that $p_\varepsilon = w$ and $p_n = 0$. This implies with (29) that $\beta^2 = p_\varepsilon = w$, so that with (44), one gets $\dot{w} = 0$. Thus, thermodynamics and the conservation laws (7c) and (7e) imply that for an equation of state $p = w\varepsilon$, the factor of proportionality w must be constant.

In order to compare equation (104) with the generalised Lifshitz-Khalatnikov theory (43), the *relativistic* analogue of (104) pertaining to the equation of state (105) will now be derived from the background equations (7) and perturbation equations (40), with $R_{(0)} = 0$ and $\Lambda = 0$. Since, by (105), one has $p_n = 0$, equations (7e) and (40d) need not be considered. With $\delta \equiv \varepsilon_{(1)}/\varepsilon_{(0)}$, equation (40b) becomes

$$\dot{\delta} + (1+w) \left[\vartheta_{(1)} + \frac{3}{2} H \delta + \frac{R_{(1)}}{4H} \right] = 0, \quad (106)$$

where $\kappa \varepsilon_{(0)} = 3H^2$, (7a), has been used. Differentiating (106) with respect to time and eliminating the time-derivatives of H , $\vartheta_{(1)}$ and $R_{(1)}$ with the help of the background equations (7) and perturbation equations (40), respectively, and, subsequently, eliminating $R_{(1)}$ with the help of (106), one finds that the system of equations (40) reduces to

$$\ddot{\delta} + 2H\dot{\delta} - \left[w \frac{\nabla^2}{a^2} + \frac{1}{2} \kappa \varepsilon_{(0)} (1+w)(1+3w) \right] \delta = -3Hw(1+w)\vartheta_{(1)}, \quad (107a)$$

$$\dot{\vartheta}_{(1)} + H(2-3w)\vartheta_{(1)} + \frac{w}{1+w} \frac{\nabla^2 \delta}{a^2} = 0, \quad (107b)$$

where it is used that $\dot{w} = 0$. The system (107) consists of two *relativistic* equations for two unknown quantities, namely the density fluctuation δ and the divergence $\vartheta_{(1)}$ of the spatial part of the fluid four-velocity. Thus, the relativistic perturbation equations (40) which are derived for a general equation of state for the pressure $p = p(n, \varepsilon)$ reduce to the relativistic system (107) for the equation of state (105) with constant w .

In the derivation of equations (40) no assumptions or approximations (other than linearisation of the Einstein equations and conservation laws) have been made. Since the relativistic system (107) is derived from the system (40), the set (107) is exact and is, therefore, superior to the standard Newtonian equation (104) which is only an approximation. As a consequence, equation (104) is only adapted to the equation of state (105) and the coefficient

$(v_s/c)^2$ must be replaced by the constant w . In other words, the equation of state $p = w\varepsilon$ requires that the speed of sound $v_s(t)$ is for all times equal to its initial value $v_s(t_0) = c\sqrt{w}$.

The gauge modes (38) are solutions of the set (40), implying that the gauge modes

$$\hat{\delta}(t, \mathbf{x}) = \frac{\psi(\mathbf{x})\dot{\varepsilon}_{(0)}(t)}{\varepsilon_{(0)}(t)} = -3H(t)\psi(\mathbf{x})(1+w), \quad \hat{\vartheta}_{(1)}(t, \mathbf{x}) = -\frac{\nabla^2\psi(\mathbf{x})}{a^2(t)}, \quad (108)$$

are solutions of equations (107), whereas $\hat{\delta}$ is a solution of (104) for constant ψ , see (64). Therefore, equations (104) and (107) have no physical significance, since the unknown gauge function $\psi(\mathbf{x})$ cannot be fixed by imposing physical initial conditions, as has been demonstrated in Section III H.

Finally, the manifestly covariant and gauge-invariant analogue of equations (107) will be given. For $R_{(0)} = 0$, $\Lambda = 0$ and an equation of state $p = w\varepsilon$ with $p_n = 0$ and constant $\beta^2 = w$, the set (43) reduces to

$$\ddot{\delta}_\varepsilon + H(1-6w)\dot{\delta}_\varepsilon - \left[w\frac{\nabla^2}{a^2} + \frac{1}{6}\kappa\varepsilon_{(0)}(1-9w)(1+3w) \right] \delta_\varepsilon = 0, \quad (109a)$$

$$\frac{1}{c}\frac{d}{dt} \left[\delta_n - \frac{\delta_\varepsilon}{1+w} \right] = 0. \quad (109b)$$

For $w = \frac{1}{3}$, these equations become equal to (68). However, for $w = 0$ the set (109) differs substantially from the set (79). This is due to the fact that, in (109), $\beta^2 = w = 0$ and $p = 0$. In contrast, in equations (79) $p \neq 0$ and $\beta(t) \approx v_s(t)/c$, (78). As a consequence, the term $\dot{\beta}/\beta = -H$ is included in (79), whereas it is missing in equations (109).

The sets (107) and (109) are both derived from the basic perturbation equations (40). The fact that the sets (107) and (109) differ considerably from each other is that (107) is based on the gauge-dependent variable $\delta \equiv \varepsilon_{(1)}/\varepsilon_{(0)}$, whereas (109) is the evolution equation for the gauge-invariant quantity $\delta_\varepsilon \equiv \varepsilon_{(1)}^{\text{gi}}/\varepsilon_{(0)}$, so that $\delta_\varepsilon \neq \delta$. As follows from (37) with $S = \varepsilon$ and (64), $\varepsilon_{(1)}$ is gauge-dependent also in the non-relativistic limit. That is why $\varepsilon_{(1)}^{\text{gi}}$ will never become equal to $\varepsilon_{(1)}$.

In both sets (107) and (109) the divergence $\vartheta_{(1)}$ of the spatial part of the fluid four-velocity is taken into account, whereas $\vartheta_{(1)}$ is missing in the standard equation (104).

The differences and similarities between equations (104), (107) and (109) will be discussed in the next two subsections. The conclusion will be that the standard equation (104) is inadequate throughout the history of the universe, whereas (107) and (109) are inadequate only after decoupling.

A. Radiation-dominated Era

In this era the fluid consists of a mixture of radiation and matter. Since radiation gives by far the largest contribution to the pressure, the equation of state for the pressure is to a very good approximation given by $p = \frac{1}{3}\varepsilon$ (i.e., $\beta^2 = w = \frac{1}{3}$) so that $p_n = 0$, implying that the right-hand side of equation (43a) vanishes and that this equation reduces to (109a). Since from the outset a realistic equation of state for the pressure $p = p(n, \varepsilon)$ has been used in the derivation of the generalised Lifshitz-Khalatnikov theory, an additional equation (43b) is found. This equation survives in the form (109b). As a consequence, the generalised Lifshitz-Khalatnikov theory predicts that perturbations in the particle number density are *gravitationally* coupled to perturbations in the radiation density, irrespective of the nature of the particles. This may prevent CDM from contracting faster than ordinary matter. This phenomenon can neither be discovered by the standard perturbation equation (104) nor by the exact equations (107), because these equations are adapted only to the equation of state (105), without taking into account the particle number density. Therefore, the equation of state (105) is incomplete.

For large-scale perturbations $\nabla^2\delta \rightarrow 0$ both (104) and the homogeneous part of (107a) have as solutions $\delta \propto t$ and $\delta \propto t^{-1}$, where the latter is the gauge mode (108). For $\nabla^2\delta \rightarrow 0$ the solution of equation (107b) is the physical mode $\vartheta_{(1)} \propto a^{-1} \propto t^{-1/2}$, implying that the corresponding particular solution of (107a) is the physical mode $\delta \propto t^{1/2}$. The physical modes of the system (107), i.e., $\delta \propto t$ and $\delta \propto t^{1/2}$, are precisely the two parts of the general solution (75) of the manifestly covariant and gauge-invariant equation (109a). This shows that the physical part of $\vartheta_{(1)}$ is contained in equation (109a). Since $\vartheta_{(1)}$ contributes only to the slow growing mode $\delta_\varepsilon \propto t^{1/2}$, $\vartheta_{(1)}$ plays only a minor role in large-scale perturbations. That is why the new perturbation theory (109) with $\vartheta_{(1)} \neq 0$ predicts the same growth rate $\delta_\varepsilon \propto t$ as the standard theory (104) which does not take $\vartheta_{(1)}$ into account.

If $\vartheta_{(1)}$ is put to zero in equation (107a) one arrives at the standard equation (104). This is not allowed, since $\vartheta_{(1)}$ is an essential quantity for the evolution of density perturbations. Indeed, if there is no fluid flow towards a

density perturbation, then that perturbation cannot grow. The generalised Lifshitz-Khalatnikov perturbation theory (109), which has $\vartheta_{(1)} \neq 0$, predicts that small-scale density perturbations, for which $\nabla^2 \delta$ is large, oscillate with an *increasing* (76) amplitude. In contrast, the standard equation (104), which has $\vartheta_{(1)} = 0$, predicts that small-scale density perturbations oscillate with a *constant* amplitude. Thus, as can be inferred from (107), $\vartheta_{(1)}$ plays a major role in the evolution of small-scale density perturbations. It is, therefore, important that the divergence of the spatial part of the fluid four-velocity $\vartheta_{(1)}$ is taken into account. Since $\vartheta_{(1)} = 0$ in the standard equation (104), this equation is inadequate to study density perturbations in the radiation-dominated era of the universe.

B. Era after Decoupling of Matter and Radiation

In this era one has $w \ll 1$, so that (104) and (107) yield nearly the same results, since $\vartheta_{(1)}$ plays only a minor role. In fact, in the limit $w \rightarrow 0$, the solution of (107b), $\vartheta_{(1)} \propto a^{-2}$, is equal to the gauge mode (108), implying that the *physical* part of $\vartheta_{(1)}$, $\vartheta_{(1)\text{physical}}$, is exactly zero. This is to be expected since $p \rightarrow 0$ implies $\vartheta_{(1)\text{physical}} \rightarrow 0$, and vice versa, see the text below (61). Therefore, equation (107b) need not be taken into account and $\vartheta_{(1)}$, being only a gauge mode, may be put to zero. In this case, the solutions of (104) and (107a) are $\delta \propto t^{2/3}$ and $\delta \propto t^{-1}$, where the latter is the gauge mode (108) with constant ψ . Since the sets (107) and (109) are both derived from the basic perturbation equations (40), the physical part of $\vartheta_{(1)}$ vanishes also in the set (109) for $w \rightarrow 0$. Therefore, the sets (107) and (109) should yield, apart from the gauge modes, exactly the same solution. Indeed, the solutions of (109a) are the modes $\delta_\varepsilon \propto t^{2/3}$ and $\delta_\varepsilon \propto t^{-1/3}$. Ignoring the decaying solutions, the three equations (104), (107) and (109) yield exactly the same result, namely *growing* density perturbations with a *scale-independent* growth rate. This result cannot be correct for the following two reasons. Firstly, since there is no fluid flow, i.e., $\vartheta_{(1)\text{physical}} \rightarrow 0$, in the limit $p \rightarrow 0$, perturbations do not grow and the gravitational field is static (65b). Secondly, given two energy density perturbations with the same energy density $\varepsilon_{(0)} + \varepsilon_{(1)}^{\text{gi}}$, then the large-scale perturbation has a stronger gravitational field than a small-scale perturbation. As a consequence, growth rates are scale-dependent. The incorrect results of equations (104), (107) and (109) are due to the equation of state (105), which is incomplete and, therefore, does not describe the cosmic fluid adequately in the perturbed universe after decoupling.

Since the generalised Lifshitz-Khalatnikov theory (43) is adapted to the realistic, non-barotropic, equation of state for the pressure $p = p(n, \varepsilon)$, it does not have the above mentioned disadvantages of the standard perturbation theory (104) and equations (107) and (109). Firstly, the equations of state (77) yield for the quantity β , (29), the correct, time-dependent, speed of sound (78), whereas $\beta^2 = w$ is constant for an equation of state $p = w\varepsilon$. Secondly, it follows from Figure 1 that in the new perturbation theory the growth rate depends on the scale of a perturbation, as it should be. There is a sharp lower limit, namely 6.2 pc, below which density perturbations do not grow at all (89).

-
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> #####
> # Relativistic Cosmological Perturbation Theory and the Evolution of Small-Scale Perturbations
> #
> # P.G.Miedema
> #####
> # Maple 16
> #
> # Consistency check of the perturbation equations (43).
> #
> # To check Eqs.(43) with coefficients (45), substitute (42) into Eqs.(43)
> # and use the background equations (7) and the perturbation equations (40)
> # to evaluate all time derivatives.
> #####
> restart;
> # Constants:
> # kappa = 8*pi*G/c^4,
> # Delta is the nabla operator,
> #####
> # Coefficients b_1, b_2, b_3, (45):
> b1:=kappa*e_0(t)*(1+w(t))/H(t)-2*diff(beta(t),t)/beta(t)-H(t)*(2+6*w(t)+3*beta(t)^2)+R_0(t)*(1/(3*H
(t))+2*H(t)*(1+3*beta(t)^2)/(R_0(t)+3*kappa*e_0(t)*(1+w(t))));


$$b1 := \frac{\kappa e_0(t) (1+w(t))}{H(t)} - \frac{2 \left( \frac{d}{dt} \beta(t) \right)}{\beta(t)} - H(t) (2+6 w(t)+3 \beta(t)^2) + R_0(t) \left( \frac{1}{3 H(t)} + \frac{2 H(t) (1+3 \beta(t)^2)}{R_0(t)+3 \kappa e_0(t) (1+w(t))} \right) \quad (1)$$

> b2:=- (1/2)*kappa*e_0(t)*(1+w(t))*(1+3*w(t))+H(t)^2*(1-3*w(t)+6*beta(t)^2*(2+3*w(t)))+6*H(t)*(diff
(beta(t),t)/beta(t))*(w(t)+kappa*e_0(t)*(1+w(t))/(R_0(t)+3*kappa*e_0(t)*(1+w(t)))-R_0(t)*((1/2)*w
(t)+H(t)^2*(1+6*w(t))*(1+3*beta(t)^2)/(R_0(t)+3*kappa*e_0(t)*(1+w(t)))-beta(t)^2*(Delta/a(t)^2-
(1/2)*R_0(t)));


$$b2 := -\frac{1}{2} \kappa e_0(t) (1+w(t)) (1+3 w(t)) + H(t)^2 (1-3 w(t)+6 \beta(t)^2 (2+3 w(t))) + \frac{6 H(t) \left( \frac{d}{dt} \beta(t) \right) \left( w(t) + \frac{\kappa e_0(t) (1+w(t))}{R_0(t)+3 \kappa e_0(t) (1+w(t))} \right)}{\beta(t)} - R_0(t) \left( \frac{1}{2} w(t) + \frac{H(t)^2 (1+6 w(t)) (1+3 \beta(t)^2)}{R_0(t)+3 \kappa e_0(t) (1+w(t))} \right) - \beta(t)^2 \left( \frac{\Delta}{a(t)^2} - \frac{1}{2} R_0(t) \right) \quad (2)$$

> b3:= (-18*H(t)^2*(e_0(t)*p_en(t)*(1+w(t))+2*p_n(t)*diff(beta(t),t)/beta(t)/(3*H(t))+p_n(t)*(p_e(t)-
beta(t)^2+n_0(t)*p_nn(t))/(R_0(t)+3*kappa*e_0(t)*(1+w(t)))+p_n(t))*(Delta/a(t)^2-(1/2)*R_0(t)) *
(n_0(t)/e_0(t)));


$$b3 := \frac{1}{e_0(t)} \left( \left( -\frac{18 H(t)^2 \left( e_0(t) p_{en}(t) (1+w(t)) + \frac{2}{3} \frac{p_n(t) \left( \frac{d}{dt} \beta(t) \right)}{\beta(t) H(t)} + p_n(t) (p_e(t) - \beta(t)^2) + n_0(t) p_{nn}(t) \right)}{R_0(t)+3 \kappa e_0(t) (1+w(t))} + p_n(t) \right) \left( \frac{\Delta}{a(t)^2} - \frac{1}{2} R_0(t) \right) n_0(t) \right) \quad (3)$$

> #####
> # Background equations
> #####
> w(t):=p_0(t)/e_0(t);


$$w(t) := \frac{p_0(t)}{e_0(t)} \quad (4)$$

> # Scale factor
> define(a,diff(a(t),t)=H(t)*a(t));
> diff(a(t),t);


$$H(t) a(t) \quad (5)$$

> # The time derivative of Eq.(7a) combined with Eqs.(7b) and (7c) yields the
> # time derivative of the Hubble function H:
> define(H,diff(H(t),t)=-(1/6)*R_0(t)-(1/2)*kappa*(e_0(t)+p_0(t)));
> diff(H(t),t);

```

$$-\frac{1}{6} R_{-0}(t) - \frac{1}{2} \kappa (e_{-0}(t) + p_{-0}(t)) \quad (6)$$

```
> # Either define the three-space curvature R_0 by Eq.(7b):
> define(R_0,diff(R_0(t),t)=-2*H(t)*R_0(t));
> # or by Eq.(8) with k=+1, 0, -1:
> # k:=+1; R_0(t):=-6*k/a(t)^2;
> diff(R_0(t),t);
```

$$-2 H(t) R_{-0}(t) \quad (7)$$

```
> # Energy conservation law (7c):
> define(e_0,diff(e_0(t),t)=-3*H(t)*e_0(t)*(1+w(t)));
> diff(e_0(t),t);
```

$$-3 H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) \quad (8)$$

```
> # Particle number conservation law (7e):
> define(n_0,diff(n_0(t),t)=-3*H(t)*n_0(t));
> diff(n_0(t),t);
```

$$-3 H(t) n_{-0}(t) \quad (9)$$

```
> # Time derivative of p_0(t):
> define(p_0,diff(p_0(t),t)=p_e(t)*diff(e_0(t),t)+p_n(t)*diff(n_0(t),t));
> diff(p_0(t),t);
```

$$-3 p_{-e}(t) H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) - 3 p_{-n}(t) H(t) n_{-0}(t) \quad (10)$$

```
> # Partial derivative p_e(t) of the pressure p:
> define(p_e,diff(p_e(t),t)=p_ee(t)*diff(e_0(t),t)+p_en(t)*diff(n_0(t),t));
> diff(p_e(t),t);
```

$$-3 p_{-ee}(t) H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) - 3 p_{-en}(t) H(t) n_{-0}(t) \quad (11)$$

```
> # Partial derivative p_n(t) of the pressure p:
> define(p_n,diff(p_n(t),t)=p_en(t)*diff(e_0(t),t)+p_nn(t)*diff(n_0(t),t));
> diff(p_n(t),t);
```

$$-3 p_{-en}(t) H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) - 3 p_{-nn}(t) H(t) n_{-0}(t) \quad (12)$$

```
> # Quantity beta(t):
> beta(t):=sqrt(diff(p_0(t),t)/diff(e_0(t),t));
```

$$\beta(t) := \frac{1}{3} \sqrt{\frac{-9 p_{-e}(t) H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) - 9 p_{-n}(t) H(t) n_{-0}(t)}{H(t) e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right)}} \quad (13)$$

```
> #####
> # First order perturbation equations
> #####
> # First order perturbation to the pressure:
> p_1(t):=p_e(t)*e_1(t)+p_n(t)*n_1(t);
```

$$p_{-1}(t) := p_{-e}(t) e_{-1}(t) + p_{-n}(t) n_{-1}(t) \quad (14)$$

```
> # Energy conservation law (40b):
> define(e_1,diff(e_1(t),t)=-3*H(t)*(e_1(t)+p_1(t))-e_0(t)*(1+w(t))*(theta(t)+(kappa*e_1(t)+(1/2)*R_1(t))/(2*H(t))));
> diff(e_1(t),t);
```

$$-3 H(t) (e_{-1}(t) + p_{-e}(t) e_{-1}(t) + p_{-n}(t) n_{-1}(t)) - e_{-0}(t) \left(1 + \frac{p_{-0}(t)}{e_{-0}(t)} \right) \left(\theta(t) + \frac{1}{2} \frac{\kappa e_{-1}(t) + \frac{1}{2} R_{-1}(t)}{H(t)} \right) \quad (15)$$

```
> # Particle number conservation law (40d):
> define(n_1,diff(n_1(t),t)=-3*H(t)*n_1(t)-n_0(t)*(theta(t)+(kappa*e_1(t)+(1/2)*R_1(t))/(2*H(t))));
> diff(n_1(t),t);
```

$$-3 H(t) n_{-I(t)} - n_{-0(t)} \left(\theta(t) + \frac{1}{2} \frac{\kappa e_{-I(t)} + \frac{1}{2} R_{-I(t)}}{H(t)} \right) \quad (16)$$

```
> # Momentum conservation law (40c):
> define(theta,diff(theta(t),t)=-H(t)*(2-3*beta(t)^2)*theta(t)-Delta/a(t)^2*p_1(t)/(e_0(t)*(1+w(t))))
;
> diff(theta(t),t);
```

$$-H(t) \left(2 + \frac{1}{3} \frac{-9 p_{-e(t)} H(t) e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right) - 9 p_{-n(t)} H(t) n_{-0(t)}}{H(t) e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right)} \right) \theta(t) - \frac{\Delta (p_{-e(t)} e_{-I(t)} + p_{-n(t)} n_{-I(t)})}{a(t)^2 e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right)} \quad (17)$$

```
> # Evolution equation for the local perturbation to the
> # global spatial curvature, (40a):
> define(R_1,diff(R_1(t),t)=-2*(H(t)*R_1(t)-kappa*e_0(t)*(1+w(t))*theta(t))-R_0(t)/(3*H(t))*(kappa*
e_1(t)+(1/2)*R_1(t)));
> diff(R_1(t),t);
```

$$-2 H(t) R_{-I(t)} + 2 \kappa e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right) \theta(t) - \frac{1}{3} \frac{R_{-0(t)} \left(\kappa e_{-I(t)} + \frac{1}{2} R_{-I(t)} \right)}{H(t)} \quad (18)$$

```
> #####
> # Gauge-invariant density perturbations
> #####
> # Gauge-invariant perturbation to the energy density, (41a):
> e_gi(t):=(e_1(t)*R_0(t)-3*e_0(t)*(1+w(t))*(2*H(t)*theta(t)+(1/2)*R_1(t)))/(R_0(t)+3*kappa*e_0(t)*
(1+w(t)));
```

$$e_{gi(t)} := \frac{e_{-I(t)} R_{-0(t)} - 3 e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right) \left(2 H(t) \theta(t) + \frac{1}{2} R_{-I(t)} \right)}{R_{-0(t)} + 3 \kappa e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right)} \quad (19)$$

```
> # Gauge-invariant perturbation to the particle number density, (41b):
> n_gi(t):=n_1(t)-3*n_0(t)*(kappa*e_1(t)+2*H(t)*theta(t)+(1/2)*R_1(t))/(R_0(t)+3*kappa*e_0(t)*(1+w(t)
));
```

$$n_{gi(t)} := n_{-I(t)} - \frac{3 n_{-0(t)} \left(\kappa e_{-I(t)} + 2 H(t) \theta(t) + \frac{1}{2} R_{-I(t)} \right)}{R_{-0(t)} + 3 \kappa e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right)} \quad (20)$$

```
> # Gauge-invariant contrast functions, (42):
> delta_e(t):=e_gi(t)/e_0(t); delta_n(t):=n_gi(t)/n_0(t);
```

$$\begin{aligned} \delta_{e(t)} &:= \frac{e_{-I(t)} R_{-0(t)} - 3 e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right) \left(2 H(t) \theta(t) + \frac{1}{2} R_{-I(t)} \right)}{\left(R_{-0(t)} + 3 \kappa e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right) \right) e_{-0(t)}} \\ \delta_{n(t)} &:= \frac{n_{-I(t)} - \frac{3 n_{-0(t)} \left(\kappa e_{-I(t)} + 2 H(t) \theta(t) + \frac{1}{2} R_{-I(t)} \right)}{R_{-0(t)} + 3 \kappa e_{-0(t)} \left(1 + \frac{p_{-0(t)}}{e_{-0(t)}} \right)}}{n_{-0(t)}} \end{aligned} \quad (21)$$

```
> # Consistency check for Eq.(43b) [left-hand side minus right-hand side should yield zero]:
> simplify(diff(delta_n(t)-delta_e(t)/(1+w(t)),t)-3*H(t)*n_0(t)*p_n(t)/(e_0(t)*(1+w(t)))*(delta_n(t)-
delta_e(t)/(1+w(t))));
```

$$0 \quad (22)$$

```
> # Consistency check for Eq.(43a) [left-hand side minus right-hand side should yield zero]:
> simplify(diff(delta_e(t),t$2)+b1*diff(delta_e(t),t)+b2*delta_e(t)-b3*(delta_n(t)-delta_e(t)/(1+w(t)
)));
```

$$0 \quad (23)$$

```
> # Conclusion: the set (43) is correct.
```

